Felipe Zingali Meira

The arithmetic and geometry of fibrations in rational and K3 surfaces

Brasil

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Tese de doutorado apresentada ao Programa de Pós-Graduação em Matemática do Instituto de Matemática da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Matemática.

Universidade Federal do Rio de Janeiro
Instituto de Matemática
&
Rijksuniversiteit Groningen
Bernoulli Institute

Orientador: Cecília Salgado
Coorientador: Jaap Top

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Stop worrying where you're going—
Move on
If you can know where you're going
You've gone
Just keep moving on
(Move On – Stephen Sondheim)

Resumo

Esta tese consiste de 3 capítulos. O primeiro capítulo lida com a introdução da teoria principal aplicada nos capítulos subsequentes.

O Capítulo 1 introduz conceitos básicos na teoria de superfícies elípticas, como sua definição, a correspondência com curvas elípticas sobre corpos de funções e a classificação de fibras singulares. Além disso, resultados específicos para superfícies elípticas racionais ou K3s são apresentados.

No Capítulo 2, estudamos o posto de uma curva elíptica $\mathcal{E}$, definida sobre o corpo de funções $k(T)$, que é dada por uma equação de Weierstrass com coeficientes de grau no máximo 2. Isto é feito estudando a fibração em cônicas e a fibração elíptica induzidas em seu modelo de Kodaira–Néron R.

No Capítulo 3, estudamos superfícies K3 X com um automorfismo não-simplético $\sigma \in \mathrm{Aut}(X)$ de ordem prima. Classificamos fibrações elípticas distintas de X com respeito à ação de σ em suas respectivas fibras. Cada tipo de fibração elíptica é relacionado a um sistema linear na resolução mínima do quociente $\tilde{R} = X/\sigma$. Quando a ação de σ no grupo de Néron–Severi de X fixa a classe da fibra de uma fibração elíptica π, este método determina quais tipos de Kodaira são admissíveis como suas fibras redutíveis. Além disso, conseguios determinar equações para sua fibra genérica.

Palavras-chaves: Superfícies elípticas, Superfícies K3.

Abstract

This thesis consists of 3 chapters. The first chapter deals with introducing the main theory used in the subsequent chapters.

Chapter 1 introduces basic concepts in the theory of elliptic surfaces, such as its main definition, the correspondence with elliptic curves over function fields and the classification of distinct fiber types. Furthermore, specific results on rational and K3 elliptic surfaces are presented.

In Chapter 2, we study the rank of an elliptic curve $\mathcal{E}$, defined over the function field $k(T)$, which is given by a Weierstrass equation with coefficients of degree at most 2. This is done by studying the induced conic and elliptic fibrations on its Kodaira–Néron model R.

In Chapter 3, we study K3 surfaces X with a non-symplectic automorphism $\sigma \in \mathrm{Aut}(X)$ of prime order. We classify distinct elliptic fibrations on X with respect to the action of σ on its respective fibers. Each type of elliptic fibrations is related to a linear system on the minimal resolution of the quocient $\tilde{R} = X/\sigma$. When the action of σ on the Néron–Severi group of X fixes the fiber class of an elliptic fibration π, this method allows us to determine which Kodaira types are admissible as its reducible fibers. Furthermore, we are able to determine equations for its generic fiber.

Key-words: Elliptic surfaces, K3 surfaces.

Samenvatting

Dit proefschrift bestaat uit drie hoofdstukken. Het eerste hoofdstuk introduceert de belangrijkste theorieën die in de daaropvolgende hoofdstukken worden gebruikt.

Hoofdstuk 1 behendelt basisbegrippen uit de theorie van elliptische oppervlakken, zoals de hoofddefinitie, het verband met elliptische krommen over functielichamen en de classificatie van verschillende vezeltypes. Daarnaast worden specifieke resultaten over rationale en K3-elliptische oppervlakken gepresenteerd.

In Hoofdstuk 2 bestuderen we de rang van een elliptische kromme $\mathcal{E}$, gedefinieerd over het functielichaam $k(T)$, die wordt gegeven door een Weierstrass-vergelijking met coëfficiënten van graad hoogstens 2. Dit doen we door de geïnduceerde vezelingen in kegelsneden en in elliptische krommen op het bij $\mathcal{E}$ behorende Kodaira–Néron-model R te bestuderen.

In Hoofdstuk 3 bestuderen we K3-oppervlakken X met een niet-symplectisch automorfisme $\sigma \in \mathrm{Aut}(X)$ van priem orde. We classificeren verschillende elliptische vezelingen op X met betrekking tot de werking van σ op de respectieve vezels. Elk type elliptische vezeling is gerelateerd aan een lineair systeem op de minimale resolutie van het quotiënt $\tilde{R} = X/\sigma$. Wanneer de werking van σ op de Néron–Severi-groep van X de vezelklasse van een elliptische vezeling π fixeert, stelt deze methode ons in staat te bepalen welke Kodaira-types kunnen voorkomen als reducibele vezels. Bovendien kunnen we vergelijkingen bepalen voor de generieke vezel van zo'n elliptische vezeling.

Sleutelwoorden: Elliptische oppervlakken, K3 oppervlakken.

Lista de ilustrações

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Notation

Let k be a perfect field with $\mathrm{char}(k) = 0$, and S a projective surface over k. We use the following notation.

$\overline{k}$ is a fixed algebraic closure of k.

$\overline{S} := S \times_k \overline{k}$ is the geometric model of S.

$\mathrm{NS}(S)$ is the Néron–Severi group of S.

$\rho(S) := \mathrm{rank}(\mathrm{NS}(S))$ is the Picard number of S.

$\chi(S)$ is the Euler characteristic of S (see (BARTH et al., 2015, Chapter I.4)).

$e(S)$ is the Euler number (or the topological Euler–Poincaré characteristic) of S (see (SCHüTT; SHIODA, 2019, Section 4.7)).

K_S is the canonical divisor of S.

$\pi \colon S \to C$ is an elliptic fibration with base C.

$F_v := \pi^{-1}(v)$ is the elliptic fiber over the point $v \in C$.

(O) is the zero-section of π.

$\mathcal{E}$ is the generic fiber of π defined over $k(C)$.

$\mathcal{E}(k(C))$ is the group of $k(C)$-points of $\mathcal{E}$ with rank r_k,

$\mathcal{E}(\overline{k}(C)) = \mathrm{MW}(\pi)$ is the group of $\overline{k}(C)$-points of $\mathcal{E}$ with rank r, which we call the Mordell–Weil group of π.

$\varphi \colon S \to C$ is a conic bundle with base C.

$G_v := \varphi^{-1}(v)$ is the conic fiber over the point $v \in C$.

Let L be a lattice and $N \subseteq L$ a sublattice.

$N^{\perp L}$ is the orthogonal complement of N inside L.

L_{root} is the root type of L, that is, the lattice generated by its roots (see (SCHüTT; SHIODA, 2019)[Definition 2.16]).

Sumário

Introduction

In the study of algebraic geometry, the dimension of a variety is one of its most important invariants. The most well understood varieties are those of dimension 1, i.e. algebraic curves. The natural next step is that of dimension 2, namely, algebraic surfaces. In comparison with curves, there are already many more open ended questions in the theory of surfaces. Fibrations appear as one way to apply the knowledge of curves to surfaces. For a projective surface S defined over a field k, a fibration on S is a surjective map $\pi\colon S \to C$, where C is a smooth projective curve over k. For each point $t \in C$, the preimage $F_t := \pi^{-1}(t)$ is called a fiber of π. The fibers of π form an infinite family of curves with strict properties on their intersection pattern (e.g. two fibers F_{t_1} and F_{t_2} do not intersect). Furthermore, if η is the generic point of the curve C, then the generic fiber F_η is a curve over $k(C)$ whose arithmetic and geometric properties are directly related to those of S.

In this work, we restrict our focus to rational and K3 surfaces. Rational surfaces, that is, surfaces which are geometrically birational to the projective plane, belong to the simplest class on the Kodaira classification, and were among the first to be investigated. On the other hand, K3 surfaces are more complicated, but are also endowed with rich geometric properties. Specifically, we study surfaces in these classes which posses more than one distinct fibration. In our first work, we study rational surfaces with an elliptic fibration and a conic bundle, i.e. fibrations in elliptic curves and conics, respectively. In our second work, we study K3 surfaces with multiple elliptic fibrations.

This thesis consists of three chapters. In the first chapter, we introduce the necessary background for what follows. We define the object in which we are mainly interested, that is, elliptic surfaces. We present some of the main results in the theory of elliptic surfaces, such as the classification of reducible fibers in a relatively minimal elliptic fibration and the Shioda–Tate formula. We define rational elliptic surfaces and present their construction as the resolution of a cubic pencil on $\mathbb{P}^2$ over an algebraically closed field. We also introduce K3 surfaces, and recall some of the results on their non-symplectic automorphisms and elliptic fibrations.

The second chapter deals with elliptic curves over a rational function field $k(T)$ over a number field k. In particular, we deal with curves $\mathcal{E}$ given by

$$y^2 = a_3(T)x^3 + a_2(T)x^2 + a_1(T)x + a_0(t), \tag{1}$$

where $a_i \in k[T]$ and $\deg a_i \leq 2$ for all a_i. Furthermore, we assume that

$$\Delta_{\text{ell}}(T) = a_3^2(-27a_0^2a_3^2 + 18a_0a_1a_2a_3 + a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3)$$

is not identically equal to 0, and $a_i(T)$ are not all multiple of the same square $(T-c)^2$.

In (NAGAO, 1997), Nagao conjectured a formula for the rank of an elliptic curve over $\mathbb{Q}(T)$. Nagao's formula consists on the limit of a weighted average of the Frobenius traces for distinct fibers $\mathcal{E}_t$, i.e. elliptic curves over $\mathbb{Q}$ obtained by the specialization map $T \mapsto t$ for each $t \in \mathbb{Q}$. In (ROSEN; SILVERMAN, 1998), Rosen and Silverman were able to prove Nagao's conjecture for elliptic curves $\mathcal{E}$ such that their Kodaira–Néron model is a rational elliptic surface. In particular, this is true for a curve $\mathcal{E}$ given by Equation 1. This fact was used to calculate the rank of several families of elliptic curves over $\mathbb{Q}(T)$ (see (ARMS; LOZANO-ROBLEDO; MILLER, 2007), (MEHRLE et al., 2017), (SADEK, 2022),(BATTISTONI; BETTIN; DELAUNAY, 2021)).

In this work, we approach the same problem (i.e. determining the rank of elliptic curves given by Equation 1) using different methods. We use two distinct geometric structures associated to the curve $\mathcal{E}$. Firstly, we use the aforementioned Kodaira–Néron model, which consists of a rational surface R together with an elliptic fibration $\pi\colon R \to \mathbb{P}^1$. Then, we notice that Equation 1) induces another kind of fibration on R; namely a conic bundle $\varphi\colon R \to \mathbb{P}^1$. We use the interaction between both geometric structures to study the rank of the elliptic curve $\mathcal{E}$ over $k(T)$.

In (ARTEBANI; GARBAGNATI; LAFACE, 2013) and (COSTA, 2024), the reducible fibers of a conic bundle on a rational elliptic surface are classified in two main types: fibers of type A_n (for $n \geq 2$) and fibers of type D_n (for $n \geq 3$). We denote the number of fibers of type A_n on $\varphi\colon R \to \mathbb{P}^1$ by δ and the rank of $\mathcal{E}$ over $\overline{k}(T)$ by r. We prove that $\delta \geq r$, and define the *defect* of $\mathcal{E}$ as $\text{Df}(\mathcal{E}) = \delta - r$ (see Definition 2.4.4). We define another important number related to $\mathcal{E}$, which we denote by δ_k, by analysing the action of $\text{Gal}(\overline{k}/k)$ on the components of the reducible fibers of φ (see 2.4.11). We show that these two numbers are sufficient for determining bounds for the rank r_k of $\mathcal{E}(k(T))$, as stated in the following.

Theorem 2.4.16. *Let r_k be the rank of $\mathcal{E}(k(T))$. Then, $\delta_k \geq r_k \geq \delta_k - \text{Df}(\mathcal{E})$.*

We show that we can determine $\text{Df}(\mathcal{E})$ using only two facts: the type of the fiber of $\varphi\colon R \to \mathbb{P}^1$ at infinity, which we call G_∞, and the Kodaira types of each fiber of $\pi\colon R \to \mathbb{P}^1$ which has a component in common with G_∞ (see 2.5.1. In particular, we use this to prove that for a general curve $\mathcal{E}$ given by Equation 1, $\mathrm{Df}(\mathcal{E}) = 0$, allowing us to conclude the following.

Theorem 2.5.2. *Let $\mathcal{E}$ be a curve given by Equation 1, and $\gamma(T) := \Delta_{ell}(T)/a_3(T)^2$. Assume that $\deg(a_3) \geq 1$, $\deg(\gamma) = 8$ and that the resultant $\mathrm{Res}(a_3, \gamma)$ is nonzero. Then, $r_k = \delta_k$.*

The third chapter deals with elliptic fibrations on K3 surfaces. One remarkable property of K3 surfaces is that they may admit several distinct elliptic fibrations. It is thus natural to classify the elliptic fibrations on a K3 surface. There are different ways of defining an equivalence between two fibrations, each leading to a different classification (details on different classifications are discussed in (BRAUN; KIMURA; WATARI, 2013)). In (OGUISO, 1989), Oguiso studied surfaces X which are Kummer surfaces of the product of two non-isogenous elliptic curves, and was able to use their geometric properties to obtain a full classification of their elliptic fibrations modulo the action from $\mathrm{Aut}(X)$. Another approach was done by Nishiyama in (NISHIYAMA, 1996), in which he used lattice theoretic techniques developed by Kneser to determine every possible ADE-type (see Definition 1.1.10) and the Mordell–Weil rank associated to an elliptic fibration on a K3 surface X with known Néron–Severi and Transcendental lattices.

Building on the works following Oguiso in (OGUISO, 1989) (see (KLOOSTERMAN, 2005) and (COMPARIN; GARBAGNATI, 2014)), Garbagnati and Salgado developed a classification method for K3 surfaces X with an involution ι which is non-symplectic, i.e. ι acts non-trivially in $H^{2,0}(X)$ (see (GARBAGNATI; SALGADO, 2019), (GARBAGNATI; SALGADO, 2020), (GARBAGNATI; SALGADO, 2024)). When the fixed locus of ι is nonempty, the quotient X/ι is a rational surface and there is a correspondence between elliptic fibration on X and linear system on X/ι. Their method consists on separating each elliptic fibration on X in three distinct types depending on the action of ι on their fibers, and deducing geometric properties of the linear systems of X/ι corresponding with elliptic fibrations of each type.

In this work, we generalize this classification to an automorphism σ with prime order $p \geq 3$. We classify each elliptic fibration $\pi\colon X \to \mathbb{P}^1$ on a K3 surface X with a non-symplectic automorphism $\sigma \in \mathrm{Aut}(X)$ with prime order as follows. We say that π is of type 1 with respect to σ if σ preserves every fiber F_v, that is, if $\sigma(F_v) = F_v$ for every $v \in \mathbb{P}^1$. We say that π is of type 2 if σ acts nontrivially on the set of fibers of π, that is, if for every fiber F_v there exists $w \in \mathbb{P}^1$ such that $\sigma(F_v) = F_w$, and for at least one $v \in \mathbb{P}^1$ $\sigma(F_v) \neq F_v$. Finally, if σ does not fix the class $F \in \mathrm{NS}(X)$ of fibers of π, then we say that π is of type 3 with respect to σ.

When $\pi\colon X \to \mathbb{P}^1$ is an elliptic fibration of type 1 with respect to σ, we are able to obtain the following constraints for the order p of σ and the Kodaira types of the singular fibers of π.

Proposition 3.3.1. *Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of prime order p. If (X, σ) admits an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ of type 1, then $p = 2$ or 3. Furthermore, if $p = 3$, the singular fibers of π must be of type I_0^*, II, IV, II^* or IV^*.*

For a K3 surface X with a non-symplectic automorphism σ of order 3, we show that each elliptic fibration $\pi\colon X \to \mathbb{P}^1$ of type 2 with respect to σ that has its zero-section preserverd by σ induces an elliptic fibration on the surface $\tilde{R}$ given as the minimal resolution of the quotient X/σ (see Proposition 3.3.9). With this, we are able to show that π comes from the base change of a rational elliptic surface (Proposition 3.3.10), a fact we exploit in order to characterize its reducible fibers.

Proposition 3.3.12. *Let $\pi_X\colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ), and assume σ preserves the zero-section. Then, σ preserves two fibers F_a^X and F_b^X, and every other fiber is in an orbit $F_{v_1}^X, F_{v_2}^X, F_{v_3}^X$ of σ. Furthermore, up to permuting F_a^X and F_b^X we have the following.*

i) F_a^X is of type I_0 or I_n^* for $n = 0, 3, 6, 9, 12$.

ii) F_b^X is of type I_0^*, III^* or I_m for $m = 0, 3, 6, 9, 12, 15, 18$.

iii) $F_{v_1}^X, F_{v_2}^X$ and $F_{v_3}^X$ have the same type, which can be II, III, IV, IV^*, I_n^* for $n = 0, 1$ or I_m for $m = 0, 1, \ldots, 6$.

With this, we are able to describe the geometric properties of the linear systems induced by any elliptic fibration $\pi\colon X \to \mathbb{P}^1$ on the rational elliptic surface $\tilde{R}$.

Theorem 3.3.15. *The induced pencil Λ is determined by the type of π.*

i) π is of type 1 if and only if Λ is a conic bundle class of $\tilde{R}$.

ii) π is of type 2 if and only if Λ is a splitting genus 1 pencil of $\tilde{R}$.

iii) π is of type 3 if and only if Λ is a non-complete linear system.

The linear system Λ is used to determine explicit equations for the generic fiber of any elliptic fibration on X of type 1 or 2 with respect to σ (see Propositions

3.3.16 and 3.3.17). Finally, these results are generalized for orders $p > 3$ under the condition that σ acts triavially on $\mathrm{NS}(X)$ (see Propositions 3.6.7, 3.6.8 and 3.6.10).

Parte I

Preparação da pesquisa

1 Background and Definitions

In this chapter, let k be a perfect field with $\operatorname{char} k = 0$. For an algebraic variety V defined over k, its geometric model is the fiber product $\overline{V} := V \times_k \overline{k}$. By the universal property of the fiber product, every map π between varieties over k induces a compatible map between their geometric models, which we denote by $\overline{\pi}$.

1.1 Elliptic Surfaces

1.1.1 Definition

Definition 1.1.1. Let S be a smooth, projective surface, and C a smooth, projective curve, both defined over k. A surjective map $\pi\colon S \to C$ is called an *elliptic fibration* if

i) all but finitely many fibers $F_v := \overline{\pi}^{-1}(v)$ for $v \in \overline{C}$, are smooth, genus 1 curves;

ii) π admits a section defined over k, i.e. a map $s\colon C \to S$ such that $\pi \circ s = \mathrm{id}_C$. We fix a section s_0 which we call the *zero-section* of π;

iii) π admits at least one singular fiber.

If S is a smooth, projective surface admitting an elliptic fibration π with section s_0, then the triple (S, π, s_0) is called an *elliptic surface*. If no fiber F_v contains a (-1)-component, we say that the fibration π is *relatively minimal*.

Remark 1.1.2. In the literature, the existence of a section is not always required. In this case, elliptic fibrations with section are called Jacobian.

Remark 1.1.3. Item (iii) in Definition 1.1.1 excludes fibrations of product type, i.e., the projection of $E \times C$ to the second coordinate, when E is an elliptic curve.

Remark 1.1.4. For simplicity, we include fibers of $\overline{\pi}$ when referring to the fibers of π. To distinguish when a fiber is specifically of π, we say that it is defined over k. We apply the same treatment to sections of $\overline{\pi}$.

Let $\mathcal{E}$ be the generic fiber of an elliptic fibration $\pi\colon S \to C$. Then, $\mathcal{E}$ is a smooth curve of genus 1 over $k(C)$. By (LANG; NéRON, 1959, Theorem 1), the groups $\mathcal{E}(k(C))$ and $\mathcal{E}(\overline{k}(C))$ are finitely generated. Let r_k and r denote their ranks, respectively.

Theorem 1.1.5. *There is a bijection between sections of π and $\overline{k}(C)$-points of $\mathcal{E}$. Furthermore, if the section is defined over k, then it corresponds to a $k(C)$-point.*

Demonstração. See (SILVERMAN, 2013, Chapter III, Proposition 3.10.(c)). □

By Theorem 1.1.5, the set of sections of $\pi\colon S \to C$ inherits the group structure of $E(\overline{k}(C))$.

Definition 1.1.6. We refer to the group of sections of $\pi\colon S \to C$ as the Mordell–Weil group of π, and denote it by $\mathrm{MW}(\pi)$.

Remark 1.1.7. The image of a section $s\colon \overline{C} \to \overline{S}$ is a curve isomorphic to $\overline{C}$ inside $\overline{S}$. We refer to the curve in $\overline{S}$ and the $\overline{k}(C)$-points of $\mathcal{E}$ induced by the same section interchangeably, distinguishing between them when necessary.

Notation 1.1.8. We denote the identity of $\mathcal{E}(\overline{k}(C))$ by O. For any $P, Q \in \mathcal{E}(\overline{k}(T))$, we denote their sum as $P \oplus Q$, the sum of P with itself n times by $[n]P$ and its respective inverse as $[-n]P$. Let $s\colon \overline{C} \to \overline{S}$ be a section corresponding to a point P. Then, we denote the curve $s(\overline{C}) \subset \overline{S}$ by (P). We assume the zero-section of π is equal to (O).

We have seen that for any elliptic surface, there is a corresponding elliptic curve over a function field. In fact, this correspondence goes both ways, as stated by the following proposition.

Theorem 1.1.9. *Let C be a smooth curve, and $k(C)$ its function field. For every elliptic curve $\mathcal{E}$ over $k(C)$ there is a unique relatively minimal elliptic surface $\pi\colon S \to C$ such that the generic fiber of π is isomorphic to $\mathcal{E}$ as an elliptic curve. This surface is called the Kodaira–Néron model of $\mathcal{E}$.*

Demonstração. See (SILVERMAN, 2013, Chapter IV, Theorem 6.1). □

1.1.2 Types of fibers on elliptic fibrations

Let F_v denote the fiber $\pi^{-1}(v)$ for $v \in \overline{C}$. Then, F_v and the zero-section (O) intersect at a single smooth point of F_v. Let $\Theta_{0,v}$ be the component of F_v which intersects (O), and let m_v be the number of distinct irreducible components of F_v. We write F_v as

$$F_v = \Theta_{0,v} + \sum_{i=1}^{m_v-1} \mu_{i,v}\Theta_{i,v},$$

where $\Theta_{i,v}$ are the distinct components of F_v and $\mu_{i,v}$ their multiplicities.

Definition 1.1.10. Let F_v be a reducible fiber of $\pi\colon S \to C$. Then, T_v is the lattice generated by the irreducible components of F_v which do not intersect (O), that is,

$$T_v := \langle \Theta_{1,v}, \ldots, \Theta_{m_v\text{-}1,v} \rangle.$$

If F_v is an irreducible fiber, then $T_v = 0$. We define the ADE-type of the fibration π as the lattice T given by the sum of T_v for each $v \in C$.

$$T := \bigoplus_{v \in \overline{C}} T_v.$$

Theorem 1.1.11. *Let F_v be a fiber on a relatively minimal elliptic fibration $\pi\colon S \to C$. Table 1 classifies every possible configuration of the components of F_v, and shows the lattice T_v, the J-function $j(F_v)$ and the Euler number $e(F_v)$.*

Demonstração. See (KODAIRA, 1963) and (NÉRON, 1964) for the classification of fibers. The values of T_v, $J(F_v)$ and $e(F_v)$ appear on (MIRANDA, 1989, Table IV.3.1). □

Theorem 1.1.12 (Tate's Algorithm)**.** *Let $\mathcal{E}$ be an elliptic curve given by a Weierstrass equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in k(C)$ for some smooth curve C. Using this equation, we can determine the Kodaira type of the fiber F_v on the Kodaira–Néron model $\pi\colon S \to C$ of $\mathcal{E}$ for each $v \in \overline{C}$.

Demonstração. See (TATE, 1975). □

The configurations of fibers in an elliptic surface is restricted by the following.

Proposition 1.1.13. *Let $\pi\colon S \to C$ be an elliptic surface. Then,*

$$e(S) = \sum_{v \in \overline{C}} e(F_v).$$

Demonstração. See (COSSEC; DOLGACHEV, 1989, Proposition 5.16). □

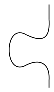
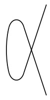
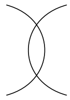
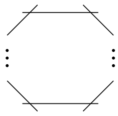
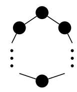
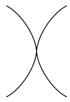
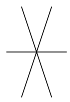
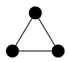
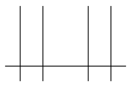
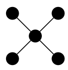
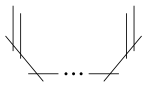
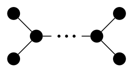
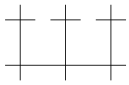
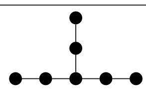
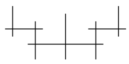
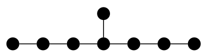
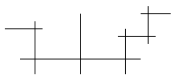
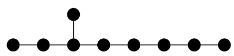
Type	Configuration	Dual graph	T_v	$J(F_v)$	$e(F_v)$
I_0		$*$	0	$\neq \infty$	0
I_1			0	∞	1
I_2			A_1	∞	2
I_n			A_{n-1}	∞	n
II			0	0	2
III			A_1	1	3
IV			A_2	0	4
I_0^*			D_4	$\neq \infty$	6
I_n^*			D_{n+4}	∞	$n+6$
IV^*			E_6	0	8
III^*			E_7	1	9
II^*			E_8	0	10

Tabela 1 – Kodaira's classification of fibers on elliptic fibrations

1.1.3 The Néron–Severi lattice

Let $\mathrm{NS}(\overline{S})$ be the group of divisors of $\overline{S}$ modulo algebraic equivalence, which we call the Néron–Severi group of $\overline{S}$ (see (SHAFAREVICH, 2013, Chapter III.4.4)). We denote the rank of $\mathrm{NS}(\overline{S})$ as a $\mathbb{Z}$-module by $\rho(\overline{S})$, and refer to it as the *Picard number* of S. For any projective surface S over k, the Néron–Severi group $\mathrm{NS}(\overline{S})$ has a bilinear pairing given by the intersection product. When S is an elliptic surface, this pairing endows $\mathrm{NS}(\overline{S})$ with a lattice structure.

Proposition 1.1.14. *Let $\pi\colon S \to C$ be an elliptic surface. Then, $\mathrm{NS}(\overline{S})$ is torsion free, and the intersection product $(\,\cdot\,)$ is non-degenerate. Consequently, $(\mathrm{NS}(\overline{S}), (\,\cdot\,))$*

has a lattice structure.

Demonstração. See (SHIODA, 1990, Theorem 1.2). □

Definition 1.1.15. Let $\pi\colon S \to C$ be an elliptic surface over k and $D \subset \overline{S}$ be an irreducible curve. We say that D is (with respect to π) *vertical* if $\overline{\pi}(D) = v \in \overline{C}$ and *horizontal* if $\overline{\pi}(D) = \overline{C}$. We say that $D \in \mathrm{NS}(\overline{S})$ is vertical (resp. horizontal) if D is represented by a sum of irreducible curves which are all vertical (resp. horizontal).

Notice that the vertical divisors in $\mathrm{NS}(\overline{S})$ correspond to fiber components of $\pi\colon S \to C$.

Definition 1.1.16. We define the trivial lattice of an elliptic surface (S, π, s_0) as

$$\mathrm{Triv}(\overline{S}) := \langle F, (O)\rangle \oplus T,$$

where F is the fiber class of π, (O) the zero-section and T the ADE-type of π (see Definition 1.1.10). Equivalently, $\mathrm{Triv}(\overline{S})$ is the sublattice of $\mathrm{NS}(\overline{S})$ generated by the prime vertical divisors of $\overline{S}$ with respect to π and the zero-section.

Theorem 1.1.17. *Let $\pi\colon S \to C$ be an elliptic fibration with generic fiber $\mathcal{E}$ defined over $k(C)$. Then, there is an isomorphism*

$$\frac{\mathrm{NS}(\overline{S})}{\mathrm{Triv}(\overline{S})} \cong \mathcal{E}(\overline{k}(C)).$$

Corollary 1.1.18. *As a consequence of Theorem 1.1.17, the Picard number of an elliptic surface is given as follows,*

$$\rho(\overline{S}) = 2 + \sum_{v \in \overline{C}} (m_v - 1) + r,$$

where r is the rank of $\mathcal{E}(\overline{k}(C))$. This is known as the Shioda–Tate formula.

Let F be the fiber class of π and (O) the zero-section. Together they determine a sublattice $\langle F, (O)\rangle \subseteq \mathrm{NS}(S)$.

Definition 1.1.19. The lattice $W_\pi := \langle F, (O)\rangle^{\perp \mathrm{NS}(S)}$ is called the *frame lattice* of (S, π, s_0).

Remark 1.1.20. Notice that on Definition 1.1.19 our notation for the orthogonal complement exhibits the ambient lattice explicitly. This is not standard, but it is useful as in Chapter 3 we work with orthogonal complements of the same sublattice over distinct ambient lattices (e.g. Tables 6 and 9).

The frame lattice is related to the ADE-type of π through the following.

Proposition 1.1.21. *Let $\pi\colon S \to C$ be an elliptic surface, and assume its Euler characteristic $\chi(S)$ is greater than 1. Then the following affirmations are true.*

i) The root of the frame lattice determines the ADE-type of π, that is, $(W_\pi)_{\mathrm{root}} = T$.

ii) $\mathrm{MW}(\pi) \cong W_\pi/T$.

Demonstração. See (SCHüTT; SHIODA, 2019, Proposition 6.42) for item (i). Item (ii) is a consequence of (i) and Theorem 1.1.17. □

The following proposition shows a connection between the linear relations of points of $\mathcal{E}(\overline{k}(T))$ and the vertical divisors of $\overline{S}$ with respect to π.

Proposition 1.1.22. *Let $P_1, \dots, P_m$ be $\overline{k}(T)$-points of the generic fiber $\mathcal{E}$ of an elliptic surface $\pi\colon S \to \mathbb{P}^1$, and let $n_1, \dots, n_m \in \mathbb{Z}$ be integers such that*

$$[n_1]P_1 \oplus \cdots \oplus [n_m]P_m = O.$$

Then for $n = n_1 + \cdots + n_m$, the divisor $n_1(P_1) + \cdots + n_m(P_m) - n(O) \in \mathrm{NS}(\overline{S})$ is vertical.

Demonstração. See (SILVERMAN, 2013, Chapter III, Proposition 9.2). □

1.1.4 Base changes of elliptic surfaces

Let $\pi\colon S \to C$ be a relatively minimal elliptic fibration. If C' is a smooth curve with a surjective map $\tau\colon C' \to C$, we can construct the fiber product $S \times_C C'$.

$$\begin{array}{ccc} S & \longleftarrow & S \times_C C' \\ \downarrow & & \downarrow \\ C & \longleftarrow & C' \end{array}$$

The generic fiber of the map $S \times_C C' \to C'$ is an elliptic curve, but $S \times_C C'$ has singularities when at least one fiber over the ramification locus of τ is singular, so it is not necessarily an elliptic surface. After resolving singularities and contracting (-1)-components of fibers, we obtain a surface S' with a relatively minimal elliptic fibration $\pi'\colon S' \to C'$. We call this elliptic surface the base change of $\pi\colon S \to C$ by $\tau\colon C' \to C$.

Take $v \in C$ and $v' \in C'$ such that $\tau(v') = v$, and let $r(v'|v)$ denote its ramification index. Let $F'_{v'}$ denote the fiber $(\pi')^{-1}(v')$ and F_v denote the fiber $\pi^{-1}(v)$. Then, it is possible to determine the Kodaira type of $F'_{v'}$ in terms of the Kodaira type of F_v and $r(v'|v)$.

Proposition 1.1.23. *The Kodaira type of $F'_{v'}$ is determined according to Table 2.*

F_v	$r(v'\|v)$	$F'_{v'}$	F_v	$r(v'\|v)$	$F'_{v'}$
I_n	m	I_{mn}	I_n^*	$m \equiv 1 \mod 2$	I_{mn}^*
				$m \equiv 0 \mod 2$	I_{mn}
II	$m \equiv 1 \mod 6$	II	II^*	$m \equiv 1 \mod 6$	II^*
	$m \equiv 2 \mod 6$	IV		$m \equiv 2 \mod 6$	IV^*
	$m \equiv 3 \mod 6$	I_0^*		$m \equiv 3 \mod 6$	I_0^*
	$m \equiv 4 \mod 6$	IV^*		$m \equiv 4 \mod 6$	IV
	$m \equiv 5 \mod 6$	II^*		$m \equiv 5 \mod 6$	II
	$m \equiv 0 \mod 6$	I_0		$m \equiv 0 \mod 6$	I_0
III	$m \equiv 1 \mod 4$	III	III^*	$m \equiv 1 \mod 4$	III^*
	$m \equiv 2 \mod 4$	I_0^*		$m \equiv 2 \mod 4$	I_0^*
	$m \equiv 3 \mod 4$	III^*		$m \equiv 3 \mod 4$	III
	$m \equiv 0 \mod 4$	I_0		$m \equiv 0 \mod 4$	I_0
IV	$m \equiv 1 \mod 3$	IV	IV^*	$m \equiv 1 \mod 3$	IV^*
	$m \equiv 2 \mod 3$	IV^*		$m \equiv 2 \mod 3$	IV
	$m \equiv 0 \mod 3$	I_0		$m \equiv 0 \mod 3$	I_0

Tabela 2 – Kodaira type of fibers after base change

Demonstração. See (MIRANDA, 1989, Table VI.4.1). □

1.2 Rational elliptic surfaces

1.2.1 Construction of rational elliptic surfaces

Definition 1.2.1. We say that R is a rational elliptic surface if R is rational and there is an elliptic fibration $\pi: R \to C$.

As a consequence of Lüroth's theorem, the basis C of π is always a rational curve. For simplicity, we assume $C = \mathbb{P}^1$.

Proposition 1.2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two cubics in $\mathbb{P}^2$ without any common components. Then, the resolution of the rational map $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $P \mapsto [\mathcal{F}(P){:}\mathcal{G}(P)]$ is a rational surface R endowed with an elliptic fibration $\pi: R \to \mathbb{P}^1$.*

Over algebraically closed fields, every relatively minimal rational elliptic surface can be obtained by such a resolution (see (MIRANDA, 1989, Lemma IV.1.2)). In general, rational elliptic surfaces have the following properties.

Proposition 1.2.3. *Let $\pi\colon R \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface defined over $\overline{k}$. The following hold.*

i) $\rho(\overline{R}) = 10$.

ii) $e(\overline{R}) = 12$.

iii) $K_R = -F$.

iv) A rational curve $C \subset R$ is a section of π if and only if $C^2 = -1$.

v) A rational curve $C \subset R$ is a fiber component of π if and only if $C^2 = -2$.

Demonstração. See (SCHüTT; SHIODA, 2019, Proposition 7.5) for items (i) and (ii), (SCHüTT; SHIODA, 2019, Proposition 5.28) for item (iii). Items (iv) and (v) are a consequence of the Adjunction Formula (see (BEAUVILLE, 1996, Theorem I.15)). □

Notice that (i) in Proposition 1.2.3 simplifies the Shioda–Tate formula (Corollary 1.1.18) when S is a rational elliptic surface, as stated by the following corollary.

Corollary 1.2.4. *Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface and r the rank of its generic fiber $\mathcal{E}$ over $\overline{k}(T)$. Then,*

$$r = 8 - \sum_{v \in \mathbb{P}^1_{\overline{k}}} (m_v - 1).$$

Let $\mathcal{E}$ be an elliptic curve over $k(T)$, and let $\pi\colon S \to \mathbb{P}^1$ be its Kodaira–Néron model. We can use a minimal Weierstrass equation for $\mathcal{E}$ to determine whether S is a rational surface.

Proposition 1.2.5. *Let $\mathcal{E}$ be given by a minimal Weierstrass equation*

$$y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x^2 + a_4(T)x + a_6(T),$$

such that $a_i \in k(T)$ are not all constant. Let $\pi\colon S \to \mathbb{P}^1$ be the Kodaira–Néron model of $\mathcal{E}$. Then, S is a rational surface if and only if $\deg(a_i) \leq i$.

Demonstração. See (SCHüTT; SHIODA, 2019, Chapter 5.13). □

1.2.2 Conic bundles on rational elliptic surfaces

Definition 1.2.6. Let S be an algebraic surface over k. A *conic bundle* on S is a surjective morphism $\varphi\colon S \to C$ to a smooth curve C such that all but finitely many fibers are irreducible curves of genus 0.

In particular, the generic fiber of a conic bundle $\varphi\colon S \to C$ is a curve of genus 0 over $k(C)$. As a consequence, we have the following proposition.

Proposition 1.2.7. *If $\varphi\colon S \to C$ is a conic bundle, then the induced morphism $\overline{S} \to \overline{C}$ has a section.*

Demonstração. By a result of (TSEN, 1933), every conic over $\overline{k}(\overline{C})$ has a point, which induces a section of φ. □

We are mainly interested in conic bundles on rational elliptic surfaces. These objects were studied in (ARTEBANI; GARBAGNATI; LAFACE, 2013) and (COSTA, 2024), for example. Let R be a rational surface and $\varphi\colon R \to C$ a conic bundle. As in the case of elliptic fibrations over rational surfaces, we assume for simplicity that $C = \mathbb{P}^1$.

Example 1.2.8. Let $\mathcal{F}$ and $\mathcal{G}$ be two smooth cubic curves in $\mathbb{P}^2$ such that $s\mathcal{F} + t\mathcal{G}$ doesn't have reducible curves and $\mathcal{F} \cap \mathcal{G}$ consists of 9 distinct points $P_1, \ldots, P_9$. Then, the induced rational elliptic surface $\pi\colon R \to \mathbb{P}^1$ has no reducible fibers. Let Λ_1 be the pencil of lines in $\mathbb{P}^2$ through P_9. Then Λ_1 induces a conic bundle $\varphi\colon R \to \mathbb{P}^1$ with exactly 8 reducible fibers, corresponding to the the lines $\ell_1, \ldots, \ell_8$, where ℓ_i is the line through P_i and P_9.

Let G be the fiber class of φ. Then, G is a nef divisor such that $G^2 = 0$ and, by adjunction, $G \cdot (-K_R) = 2$. In what follows, we see that these conditions are sufficient for characterizing the conic bundles of R.

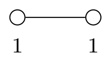
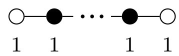
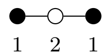
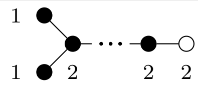
Definition 1.2.9. Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface. Then, $G \in \mathrm{NS}(\overline{R})$ is called a *conic class* if G is nef, $G^2 = 0$ and $G \cdot (-K_R) = 2$.

Proposition 1.2.10. *Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface. Then, every conic class G induces a conic bundle $\varphi\colon R \to \mathbb{P}^1$ with G as its fiber class.*

Demonstração. See (COSTA, 2024, Theorem 3.8). □

Assume that the elliptic fibration $\pi\colon R \to \mathbb{P}^1$ is relatively minimal. Then, we classify the fibers of φ as follows.

Theorem 1.2.11. *Let R be a relatively minimal rational elliptic surface and $\varphi\colon R \to \mathbb{P}^1$ a conic bundle. Then, every fiber of φ fits in one of the types described in Table 3.*

Type	Intersection Graph
0	$*$
A_2	
A_n $(n \geq 3)$	
D_3	
D_n $(n \geq 4)$	

$*$ smooth, irreducible curve of genus zero

○ (-1)-curve (section of π)

● (-2)-curve (component of a reducible fiber of π)

Tabela 3 – fibers in conic bundles over rational elliptic surfaces

Demonstração. See (COSTA, 2024, Theorem 4.2). □

Example 1.2.12. Let $\varphi\colon R \to \mathbb{P}^1$ be the conic bundle described in Example 1.2.8, and let $G_1, \ldots, G_8$ be its reducible fibers. Since the elliptic fibration $\pi\colon R \to \mathbb{P}^1$ has no reducible fibers, by Proposition 1.2.3 R has no rational (-2)-curves. Then, by Theorem 1.2.11, every G_i is a fiber of type A_2. The components of G_i correspond to $\tilde{\ell}_i$ and E_i, which are the strict transform of ℓ_i and the exceptional curve of the blow-up of P_i, respectively.

1.2.3 Splitting genus 1 pencils

Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface. We are interested in pencils of curves of genus 1 on R. One clear example of such system is the pencil of fibers induced by π. When π is relatively minimal, this is the only base point free pencil of genus 1 curves. On the other hand, if π is not relatively minimal, it is possible to have multiple such pencils.

Definition 1.2.13. A *splitting genus 1 pencil* on a rational elliptic surface (which may not be relatively minimal) $\tilde{R}$ is a proper morphism $\varphi\colon \tilde{R} \to \mathbb{P}^1$ such that

i) $C_s := \varphi^{-1}(s)$ is a genus 1 curve for almost all $s \in \mathbb{P}^1$.

ii) $C_s \cdot K_{\tilde{R}} = 0$ for all $s \in \mathbb{P}^1$.

The pencil of curves $\{C_s\}_{s\in\mathbb{P}^1}$ is also called a splitting genus 1 pencil.

The use of the term "splitting"comes from the fact that under specific base changes, each splitting genus 1 pencil $\varphi\colon \tilde{R} \to \mathbb{P}^1$ determines an elliptic fibrations on a K3 surface X such that every genus 1 curve C_s is split into a fixed number of isomorphic copies (see Theorem 3.3.15).

Example 1.2.14. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be cubics in $\mathbb{P}^2$ with no common components, and assume that $\mathcal{H}$ is not in the pencil $\Lambda_1 = \mathcal{F} + t\mathcal{G}$. We can define a pencil $\Lambda_2 = \mathcal{F} + t\mathcal{H}$ distinct from Λ_1. Let $\tilde{R}$ be the surface obtained by blowing up the base points of both Λ_1 and Λ_2. These pencils induce two distinct elliptic fibrations $\pi_1\colon \tilde{R} \to \mathbb{P}^1$ and $\pi_2\colon \tilde{R} \to \mathbb{P}^1$ which are not relatively minimal, and the families $\{\pi_i^{-1}(s)\}_{s\in\mathbb{P}^1}$ are splitting genus 1 pencils of $\tilde{R}$.

1.3 K3 surfaces

In this section, we assume that k is an algebraically closed field, that is, $\overline{k} = k$, and $\mathrm{char}(k) = 0$. As a consequence, for every variety V we have that $V = \overline{V}$.

1.3.1 Definition

Definition 1.3.1. A *K3 surface* is a smooth, projective surface X over k with trivial canonical divisor and irregularity 0, that is, $K_X = 0$ and $q(X) = h^1(X, \mathcal{O}_X) = 0$.

Example 1.3.2. Let $\pi\colon X \to \mathbb{P}^2$ be a double covering ramified over a smooth sextic C. Then, X is a K3 surface (see (HUYBRECHTS, 2016, Chapter 1, Example 1.3.(iv))).

We can calculate the following for K3 surface.

Proposition 1.3.3. *Let X be a K3 surface. Let $\chi(X)$ denote the Euler characteristic and $e(X)$ the Euler number (see (SCHüTT; SHIODA, 2019, Section 4.7)) of X. The following is true.*

i) $\chi(X) = 2$.

ii) $e(X) = 24$.

iii) $\rho(X) \leq 20$.

Remark 1.3.4. Let k be a field with $\text{char}(k) > 0$. Then, item (iii) in Proposition 1.3.3 is not true for every K3 surface. Instead, there is a bound $\rho(X) \leq 22$ (see (HUYBRECHTS, 2016)[Chapter 1, Remark 3.7]), and there are known for which $\rho(X) = 22$ (see (SHIODA, 1973)).

Demonstração. See (HUYBRECHTS, 2016, Chapter 1, 2.3) for item (i). Item (ii) is a consequence of (i) and Noether's Formula (see (BEAUVILLE, 1996, I.14)), and Item (iii) is proved in (HUYBRECHTS, 2016, Chapter 1, 3.3). □

Proposition 1.3.5. *The lattice $H^2(X, \mathbb{Z})$ endowed with the cup product is isometric to $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, which we call the K3 lattice.*

Demonstração. See (HUYBRECHTS, 2016, Chapter 1, Proposition 3.5). □

The intersection product of X endows $\text{NS}(X)$ with a lattice structure (see (HUYBRECHTS, 2016, Chapter 1, Proposition 2.4)), which can be isometrically embedded in $H^2(X, \mathbb{Z})$ via the first Chern map. We consider its orthogonal complement in the following definition.

Definition 1.3.6. The transcendental lattice of X is defined as

$$T_X := \text{NS}(X)^{\perp H^2(X,\mathbb{Z})}.$$

1.3.2 Automorphisms of K3 surfaces

Let X be a K3 surface and $\sigma \in \text{Aut}(X)$ an automorphism of X. Since $K_X = 0$, we have $h^0(X, K_X) = h^0(X, \mathcal{O}_X) = 1$, and we can choose a generator ω_X of $H^0(X, K_X)$. Then, σ induces an action on $H^0(X, K_X)$ with the map $\omega_X \mapsto \sigma^*(\omega_X)$.

Definition 1.3.7. We say that an automorphism σ is *symplectic* if $\sigma^*(\omega_X) = \omega_X$ and *non-symplectic* otherwise, that is, if $\sigma^*(\omega_X) = \alpha\omega_X$ for some $\alpha \in k$.

Theorem 1.3.8. *Let X be a K3 surface and σ a non-symplectic automorphism of finite order m. Then, $\sigma^*(\omega_X) = \zeta_m\omega_X$, where $1 \neq \zeta_m$ is an mth root of unity. Furthermore, $\varphi(m) \leq 21$, and consequently $m \leq 66$.*

Demonstração. See (NIKULIN, 1980, Theorem 0.1). □

Let X be a K3 surface endowed with a non-symplectic automorphism σ of prime order p. By Theorem 1.3.8, p is at most equal to 19. Denote the fixed locus of σ by $\text{Fix}(\sigma)$.

Theorem 1.3.9. *Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of prime order p. If $p = 2$, then $\mathrm{Fix}(\sigma)$ is either empty, the disjoint union of two elliptic curves or the disjoint union of m smooth curves $C \sqcup L_1 \sqcup \ldots \sqcup L_{m\text{-}1}$, where C is a curve of genus $g \geq 0$ and L_i are rational curves. If $p \geq 3$, then*

$$\mathrm{Fix}(\sigma) = C \sqcup L_1 \sqcup \ldots \sqcup L_{m-1} \sqcup \{P_1, ..., P_n\},$$

where C is a smooth curve of genus $g \geq 0$, L_i are smooth rational curves and P_i isolated fixed points.

Demonstração. See (ARTEBANI; SARTI; TAKI, 2011, Lemma 2.2). □

Proposition 1.3.10. *Let $x \in X$ be a point fixed by a non-symplectic automorphism σ of order p. Then there are local coordinates (z_1, z_2) around x such that the action of σ is given by one of the following linear maps*

$$A_{p,t} = \begin{pmatrix} \zeta_p^{t+1} & 0 \\ 0 & \zeta_p^{p-t} \end{pmatrix}, t = 0, 1, ..., p-2,$$

where ζ_p is a primitive root of unity of order p. If the local action is given by $A_{p,0}$, then x is a point in a fixed curve. Otherwise, x is an isolated fixed point.

Demonstração. The fact that an automorphism can be seen to act linearly around a fixed point is a classical result by Cartan (see (CARTAN, 1954-1954, Lemma 1)). The application to non-symplectic automorphisms stems from work of Nikulin (see (NIKULIN, 1980, Section 5)). □

Remark 1.3.11. For a non-symplectic involution, the only possible action around a fixed point is $(z_1, z_2) \mapsto (-z_1, z_2)$, which indicates that the fixed point is part of a fixed curve. In other words, there are no isolated fixed points.

Theorem 1.3.12. *Let X be a K3 surface and $\sigma \in \mathrm{Aut}\, X$ of finite order $n > 1$. Let Y be the resolution of the quotient X/σ.*

i) If σ is symplectic, then X/σ is a K3 surface.

ii) If σ is non-symplectic with $n = 2$ and $\mathrm{Fix}(\sigma) = \varnothing$, then Y is an Enriques surface.

iii) If σ is non-symplectic and either $n > 2$ or $\mathrm{Fix}(\sigma) \neq \varnothing$, then Y is rational.

Demonstração. See (KONDō, 2018, Lemma 4.1) for item (i), (ZHANG, 1999, Lemma 1.2), and (XIAO, 1995, Lemma 2) for items (ii) and (iii). □

1.3.3 Elliptic fibrations on K3 surfaces

Let X be a K3 surface and $\pi\colon X \to C$ a relatively minimal elliptic fibration. As in the case of rational elliptic surfaces, we assume $C = \mathbb{P}^1$. Let F be the fiber class of π and (O) the zero-section. The lattice $\langle F, F + (O)\rangle$ determines an embedding of the hyperbolic lattice U into $\mathrm{NS}(X)$. On the other hand, the following holds.

Proposition 1.3.13. *Let X be a K3 surface and $\psi\colon U \hookrightarrow \mathrm{NS}(X)$ an embedding of lattices. Then, there is a relatively minimal elliptic fibration $\pi\colon X \to \mathbb{P}^1$ such that its frame lattice W_π is isometric to $\psi(U)^{\perp \mathrm{NS}(X)}$.*

Demonstração. See (KONDō, 1992, Lemma 2.1). □

Since U is unimodular, we can write $\mathrm{NS}(X) = U \oplus W_\pi$. Consequently, W_π is an even lattice with signature $(0, \rho(X) - 2)$ and its discriminant lattice is isomorphic to that of $\mathrm{NS}(X)$.

Proposition 1.3.14. *Let $\pi\colon S \to \mathbb{P}^1$ be an elliptic fibration. Then, S is a K3 surface if and only if $e(S) = 24$.*

Demonstração. If S is a K3 surface, then $e(S) = 24$ by Theorem 1.3.3. Assume $e(S) = 24$. Since S is elliptic, we can write its canonical divisor as

$$K_S = (\chi(S) - 2)F, \tag{1.1}$$

where F is the fiber class of π and $\chi(S)$ the Euler characteristic of S. In particular $K_S^2 = 0$. Thus, by Noether's Formula (see (BEAUVILLE, 1996, I.14)), we have

$$\chi(S) = \frac{e(S) + K_S^2}{12} = 2.$$

Consequently, by Equation 1.1 we have $K_S = 0$. By Serre Duality ((BEAUVILLE, 1996, Theorem I.11)), $h^2(S, \mathcal{O}_S) = h^0(S, K_S) = h^0(S, \mathcal{O}_S) = 1$. Thus, by the definition of the Euler characteristic, we have

$$q(S) = h^1(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) - \chi(S) = 0.$$

This proves that S is a K3 surface. □

Parte II

Resultados

2 Conic bundles and Mordell–Weil ranks of elliptic surfaces

2.1 Introduction

This chapter is based on the paper (MEIRA, 2025). This paper is accessible through arXiv and is currently under review for publication.

Let k be a number field, and $\mathcal{E}$ a curve over the function field $k(T)$ given by an equation of the form

$$y^2 = a_3(T)x^3 + a_2(T)x^2 + a_1(T)x + a_0(t), \tag{2.1}$$

where $a_i(T)$ are polynomials in $k[T]$ of degree at most 2. We further assume that

$$\Delta_{\text{ell}}(T) = a_3^2(-27a_0^2a_3^2 + 18a_0a_1a_2a_3 + a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3)$$

is not identically equal to 0, and $a_i(T)$ are not all multiple of the same square $(T-c)^2$. With these conditions, $\mathcal{E}$ defines an elliptic curve over $k(T)$. Curves in this form have been studied in (ARMS; LOZANO-ROBLEDO; MILLER, 2007), (KOLLAR; MELLA, 2017), (BATTISTONI; BETTIN; DELAUNAY, 2021).

In this chapter, we study the Mordell–Weil rank of r_k of $\mathcal{E}(k(T))$ through the geometry determined by Equation 2.1. By Theorem 1.1.9, $\mathcal{E}$ has a Kodaira–Néron model $\pi\colon R \to \mathbb{P}^1$. By applying the changes of coordinates $x \mapsto \frac{x}{a_3(T)}$ and $y \mapsto \frac{y}{a_3(T)}$, we can use Proposition 1.2.5 to deduce that R is rational. Furthermore, we can rewrite Equation 2.1 as

$$y^2 = A(x)T^2 + B(x)T + C(x). \tag{2.2}$$

The projection to the x-coordinate in Equation 2.2 determines a conic bundle $\varphi\colon R \to \mathbb{P}^1$. Each root θ of $\Delta_{\text{conic}}(x) := B(x)^2 - 4A(x)C(x)$ induces a $\overline{k}(T)$-point $P_\theta \in \mathcal{E}(\overline{k}(T))$. Namely,

$$P_\theta := \begin{cases} \left(\theta, \sqrt{A(\theta)}\left(T + \frac{B(\theta)}{2A(\theta)}\right)\right) & \text{when } A(\theta) \neq 0, \\ \left(\theta, \sqrt{C(\theta)}\right) & \text{when } A(\theta) = 0. \end{cases} \tag{2.3}$$

Each of these points corresponds to a (-1)-component in a reducible fiber of φ (see Theorem 1.2.11). As an application of Nagao's conjecture ((NAGAO, 1997)), which was proven by Rosen and Silverman for rational elliptic surfaces ((ROSEN; SILVERMAN, 1998)), we obtain families of curves $\mathcal{E}$ in which the number of $k(T)$-points induced by roots of $\Delta_{\text{conic}}(x)$ is equal to the rank r_k exactly (see (ARMS; LOZANO-ROBLEDO; MILLER, 2007, Theorem 2.1)). This is not true in general – indeed, there are cases where r_k is strictly smaller than the number of roots of Δ_{conic} (see (SHIODA, 1991, Theorem A_2)).

2.1.1 Chapter structure

Section 2.2 surveys Nagao's conjecture and its applications to determine the Mordell–Weil rank of elliptic curves as in Equation (2.1).

Section 2.3 deals with a general rational surface R with a relatively minimal elliptic fibration $\pi\colon R \to \mathbb{P}^1$ and a conic bundle $\varphi\colon R \to \mathbb{P}^1$. In 2.3.1, we adapt results from standard conic bundles to obtain a description of the Néron–Severi group $\mathrm{NS}(\overline{R})$ and the canonical divisor K_R. In 2.3.2, we compare the number δ of fibers of type A_n in φ and the rank r of the generic fiber of π.

In Section 2.4 we apply the results of the previous section to surfaces R given by Equation (2.1) and Equation (2.2). In 2.4.1, we determine the types of the singular fibers of the conic bundle φ from Equation (2.2). In 2.4.2, we define the defect of $\mathcal{E}$, and prove that the points P_θ induced by the conic bundle φ generate a finite index subgroup of $\mathcal{E}(\overline{k}(T))$. In 2.4.3 we define the number δ_k in terms of the action of $\mathrm{Gal}(\overline{k}/k)$ on the fibers of φ and prove our main result, $\delta_k - \mathrm{Df}(\mathcal{E}) \leq r_k \leq \delta_k$.

Section 2.5 deals with using the bounds for r_k to determine the rank r_k of families of curves given by Equation (2.1). In 2.5.1, we determine $\mathrm{Df}(\mathcal{E})$ from the fiber configuration of π and φ. In 2.5.2, we explore cases in which $\mathrm{Df}(\mathcal{E}) = 0$, and provide families for which $r_k = \delta_k$. In 2.5.3 we explore cases with $\mathrm{Df}(\mathcal{E}) > 0$, and provide families with $\mathrm{Df}(\mathcal{E}) = 1$ for which we can determine if $r_k = \delta_k$ or $r_k = \delta_k - 1$ depending on the coefficients of $\mathcal{E}$ in Equation (2.2).

2.2 Nagao's Conjecture and Applications

In this section, we state Nagao's conjecture and give a brief exposition on subsequent theorems and applications. We follow the original exposition of the conjecture in (NAGAO, 1997), so we work over $\mathbb{Q}$.

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{Q}(T)$, given by an equation in Weierstrass form

$$\mathcal{E}:\ y^2 + a_1(T)xy + a_3(T)y = x^3 + a_2(T)x + a_4(T)x + a_6(T),$$

with coefficients $a_i(T) \in \mathbb{Z}[T]$. For all but finitely many $t \in \mathbb{Q}$, the specialization map $T \mapsto t$ yields an elliptic curve $\mathcal{E}_t$ over $\mathbb{Q}$. For each prime $p \in \mathbb{Z}_{>0}$ of good reduction, we consider $a_t(p)$ the trace of the Frobenius at p on $\mathcal{E}_t$, given by $p+1-N_t(p)$, where $N_t(p)$ is the number of $\mathbb{F}_p$ points of $\mathcal{E}_t$ (we say that $a_t(p) = 0$ if $p \mid \Delta(t)$). Further, consider the average over fibers

$$A_{\mathcal{E}}(p) := \frac{1}{p}\sum_{t=0}^{p-1} a_t(p).$$

In 1997, based on several explicit calculations for the Mordell–Weil rank of $\mathcal{E}(\mathbb{Q}(T))$ on nontrivial families, Nagao conjectured a limit formula relating $\operatorname{rank}\mathcal{E}(\mathbb{Q}(T))$ to the values of $A_{\mathcal{E}}(p)$ (see (NAGAO, 1997, Question (2))).

Conjecture 2.2.1. *Let $\mathcal{E}$, $A_{\mathcal{E}}(p)$ be as defined above, then*

$$\lim_{X\to\infty} \frac{1}{X}\sum_{p\le X} -A_{\mathcal{E}}(p)\log p = \operatorname{rank}\mathcal{E}(\mathbb{Q}(T)).$$

In the following year, Rosen and Silverman published a proof that Tate's conjecture implies Nagao's conjecture, settling it in particular for rational elliptic surfaces (see (ROSEN; SILVERMAN, 1998, Theorem 1.3)).

Theorem 2.2.2 (Rosen, Silverman)**.** *Nagao's conjecture holds for rational elliptic surfaces.*

In (ARMS; LOZANO-ROBLEDO; MILLER, 2007), Arms, Lozano-Robledo and Miller apply Rosen and Silverman's result to elliptic curves over $\mathbb{Q}(T)$ given by an equation of the form

$$\begin{aligned} y^2 &= x^3T^2 + 2g(x)T - h(x), \text{ where} \\ g(x) &= x^3 + ax^2 + bx + c,\ c \neq 0; \\ h(x) &= (A-1)x^3 + Bx^2 + Cx + D. \end{aligned} \tag{2.4}$$

Applying the coordinate change $x \mapsto \frac{x}{T^2+2T-A+1}$ to Equation 2.4, we check that it indeed corresponds to a rational elliptic surface (see (SCHüTT; SHIODA, 2019, Chapter 5.13)). The theory of quadratic Legendre sums is used to prove the following.

Theorem 2.2.3. *For infinitely many integers a, b, c, A, B, C, D, the polynomial $D_T(x) = g(x)^2 + x^3 h(x)$ has 6 distinct, nonzero roots which are perfect squares, and the curve $\mathcal{E}$ given by Equation 2.4 is an elliptic curve over $\mathbb{Q}(T)$ with $\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) = 6$.*

Demonstração. See (ARMS; LOZANO-ROBLEDO; MILLER, 2007, Theorem 2.1 and Remark 2.2). □

This was later generalized to any number field in (MEHRLE et al., 2017, Theorem 1.1). In (SADEK, 2022, Theorem 5.3), this result is expanded to curves over $\mathbb{Q}(T)$ given by Equation 2.1 such that $\deg a_3(T) = 2$.

A similar strategy was employed by Battistoni, Bettin and Delaunay in (BATTISTONI; BETTIN; DELAUNAY, 2021) to obtain rank formulas for elliptic curves over $\mathbb{Q}(T)$ of the form

$$y^2 = A(x)T^2 + B(x)T + C(x), \tag{2.5}$$

where $\deg A(x), \deg B(x) \leq 2$, at least one of $A(x)$, $B(x)$ is not identically zero, and $C(x)$ is monic and of degree 3. They obtain four distinct formulas for $\operatorname{rank} \mathcal{E}(\mathbb{Q}(T))$, depending on if $A(x) = 0$, $A(x) = \mu \in \mathbb{Q}^\times$, $\deg(A) = 1$ or $\deg(A) = 2$ (see (BATTISTONI; BETTIN; DELAUNAY, 2021, Theorem 1)).

When $A(x) = 0$, the formula is simplified as follows.

Theorem 2.2.4 (Battistoni, Bettin, Delaunay)**.** *Let $\mathcal{E}$ be an elliptic curve over $\mathbb{Q}(T)$ given by an equation of the form 2.5, and assume $A(x) = 0$. Then,*

$$\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) = \#\{[\theta] : B(\theta) = 0, C(\theta) \in \mathbb{Q}(\theta)^2 \setminus \{0\}\},$$

where $[\theta]$ denotes the orbit of θ by the action of the Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Demonstração. See (BATTISTONI; BETTIN; DELAUNAY, 2021, Theorem 2). □

When $A(x) = \mu \in \mathbb{Q}^\times$, the formula is also simplified.

Theorem 2.2.5 (Battistoni, Bettin, Delaunay)**.** *Let $\mathcal{E}$ be an elliptic curve over $\mathbb{Q}(T)$ given by an equation of the form 2.5, and assume $A(x) = \mu \in \mathbb{Q}^\times$. Then,*

$$\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) = \begin{cases} \#\{[\theta] : B^2(\theta) - 4\mu C(\theta) = 0\} - 1 & \text{if } \mu \in \mathbb{Q}^2, \\ \#\{[\theta] : B^2(\theta) - 4\mu C(\theta) = 0, \mu \in \mathbb{Q}(\theta)^2 \setminus \{0\}\} & \text{if } \mu \in \mathbb{Q} \setminus \mathbb{Q}^2. \end{cases}$$

Demonstração. See (BATTISTONI; BETTIN; DELAUNAY, 2021, Theorem 1, Section 3.1). □

When $\deg(A) = 1$ or 2, the formula provides the following upper bound for the rank.

Theorem 2.2.6 (Battistoni, Bettin, Delaunay)**.** *Let $\mathcal{E}$ be an elliptic curve over $\mathbb{Q}(T)$ given by an equation of the form 2.5. Then,*

$$\operatorname{rank} \mathcal{E}(\mathbb{Q}(T)) \leq \#\{[\theta] : B^2(\theta) - 4A(\theta)C(\theta) = 0\}.$$

Demonstração. See (BATTISTONI; BETTIN; DELAUNAY, 2021, Proposition 14). □

2.3 Conic bundles on rational elliptic surfaces

In this section, let R be a rational elliptic surface with a relatively minimal elliptic fibration $\pi\colon R \to \mathbb{P}^1$, and $\varphi\colon R \to \mathbb{P}^1$ a conic bundle.

2.3.1 Generalities on conic bundles

We start by establishing notation for the reducible fibers of the conic bundle $\varphi\colon R \to \mathbb{P}^1$. By Theorem 1.2.11, every reducible fiber of φ is of type A_n, with $n \geq 2$, or D_n, with $n \geq 3$. Let $\delta(\varphi)$ be the number of fibers of type A_n, and $\varepsilon(\varphi)$ the number of fibers of type D_n. Notice that these numbers depend on the conic bundle φ; a rational elliptic surface may be endowed with two different conic bundles φ_1 and φ_2 such that $\delta(\varphi_1) \neq \delta(\varphi_2)$. Through the rest of this section, we fix one conic bundle φ and refer to $\delta(\varphi), \varepsilon(\varphi)$ as δ, ε, respectively. Next, we establish further notation for the fibers of φ.

Notation 2.3.1. A fiber $\varphi^{-1}(v)$ is denoted by G_v, its number of components by n_v and its class in $\mathrm{NS}(\overline{R})$ by G.

Denote the fibers of type A_n by $G_{v_1}, \ldots, G_{v_\delta}$. We write

$$G_{v_i} = \sum_{j=0}^{n_{v_i}\text{-}1} \alpha_{v_i,j}. \tag{2.6}$$

The components in Equation 2.6 intersect following the graph of fibers of type A_n in Table 3, with $\alpha_{v_i,0}^2 = \alpha_{v_i,n_{v_i}\text{-}1}^2 = -1$ and $\alpha_{v_i,j}^2 = -2$ for $j = 1, \ldots, n_{v_i} - 2$.

Denote the fibers of type D_n by $G_{w_1}, \ldots, G_{w_\varepsilon}$. We write

$$G_{w_i} = \beta_{w_i,0} + \beta_{w_i,1} + 2 \sum_{j=2}^{n_{w_i}\text{-}1} \beta_{w_i,j}. \tag{2.7}$$

Similarly, the components in Equation 2.7 intersect following the graph of fibers of type D_n in Table 3, with $\beta^2_{w_i,n_{w_i}\text{-}1} = -1$ and $\beta^2_{w_i,j} = -2$ for $j = 0, \ldots, n_{w_i} - 2$.

Let G_v be a reducible fiber of φ. Then $e(G_v) = n_v + 1$ independently of the type of G_v, since the fiber is composed of n_v rational curves intersecting in $n_v - 1$ distinct points. We can use this fact to limit the possible configurations of fibers.

Proposition 2.3.2. *For $\varphi\colon R \to \mathbb{P}^1$ a conic bundle over a rational elliptic surface, we have the following formula.*

$$\sum_{v \in \mathbb{P}^1} (n_v - 1) = 8.$$

Demonstração. By (COSSEC; DOLGACHEV, 1989, Proposition 5.1.6), the Euler number of R is given by

$$e(\overline{R}) = e(G_\eta)e(\mathbb{P}^1) + \sum_{v \in \mathbb{P}^1} (e(G_v) - e(G_\eta)),$$

where G_η is the generic fiber of φ. By Proposition 1.2.3, $e(\overline{R}) = 12$. Substituting $e(G_\eta) = e(\mathbb{P}^1) = 2$ and $e(G_v) = n_v + 1$, we obtain the result. □

In what follows, we deal with the concept of standard conic bundles, which is defined as follows.

Definition 2.3.3. Let $\varphi\colon S \to \mathbb{P}^1$ be a conic bundle. We say that it is *standard* if every reducible fiber of φ is given by two concurrent rational (-1)-curves.

Notice that, if $\varphi\colon R \to \mathbb{P}^1$ is a conic bundle and R is a rational elliptic surface with a relatively minimal elliptic fibration $\pi\colon R \to \mathbb{P}^1$, then φ is standard if and only if every reducible fiber is of type A_2 with respect to the classification in Theorem 1.2.11.

When a rational surface R has a standard conic bundle, we can describe the generators of the Néron–Severi group $\mathrm{NS}(\overline{R})$, and exhibit the canonical divisor K_R explicitly in terms of said generators.

Theorem 2.3.4. *Let S be a rational surface such that $K_S^2 = d$, and let $\varphi\colon S \to \mathbb{P}^1$ be a standard conic bundle. The following hold.*

i) There are $r' = 8 - d$ reducible fibers of φ, all of which are composed of two concurrent exceptional curves.

ii) There is a free basis of $\mathrm{NS}(\overline{S})$ given by $\langle G, H, \ell_1, ..., \ell_{r'} \rangle$, where G is the fiber class of φ, H is a section of φ and $\ell_1, ..., \ell_{r'}$ are the components of the reducible fibers of φ not intersecting H.

iii) The canonical divisor of S is given by

$$K_S = -2H + (H^2 - 2)G + \textstyle\sum_{i=1}^{r'} \ell_i.$$

Demonstração. See (KUNYAVSKII; TSFASMAN, 1985, Proposition 0.4). □

The rest of this section is devoted to generalizing Theorem 2.3.4 to any conic bundle over a (relatively minimal) rational elliptic surface.

Assumption 2.3.5. Fix a section $H \subset R$ of the conic bundle $\varphi\colon R \to \mathbb{P}^1$. Then, $H \cdot G_v = 1$, so H intersects a single simple component of G_v. For the fibers G_{w_i} of type D_n, H can only intersect $\beta_{w_i,0}$ or $\beta_{w_i,1}$, and we can assume without loss of generality that it intersects $\beta_{w_i,0}$. On the other hand, for the fibers G_{v_i} of type A_n, H can intersect any component. Let k_i be the number such that H intersects α_{v_i,k_i}. We can assume without loss of generality that $0 \leq k_i \leq n_{v_i} - 2$.

Proposition 2.3.6. *Let R be a rational elliptic surface and $\varphi\colon R \to \mathbb{P}^1$ a conic bundle on R with a fixed section $H \subset R$. Then, there is a contraction $\eta\colon R \to R_0$ such that*

i) H does not intersect any of the curves contracted by η;

ii) there is a standard conic bundle $\varphi_0\colon R_0 \to \mathbb{P}^1$ such that $\varphi = \varphi_0 \circ \eta$.

In other words, $\eta(H)$ is a section of $\varphi_0\colon R_0 \to \mathbb{P}^1$.

Demonstração. Let E be a (-1)-component of a fiber of φ. Then, the pushforward of the fiber class G by the blow-down of E induces a conic bundle commuting with the blow-down map.

Let G_{v_i} be a fiber of type A_n such that $n_{v_i} \geq 3$. If we blow-down one of the (-1)-components of G_{v_i}, the (-2)-component intersecting it becomes a (-1)-component on the contracted surface. Thus, we can repeat this process successively. We need to contract $n_{v_i} - 2$ components that do not intersect the section H. Since we assume that H intersects α_{v_i,k_i}, we can do this by blowing-down a chain of k_i components starting with $\alpha_{v_i,0}$ and a chain of $n_{v_i} - k_i - 2$ components starting with $\alpha_{v_i,n_{v_i}\text{-}1}$. This iterative process yields a fiber of type A_2 (see Figure 1).

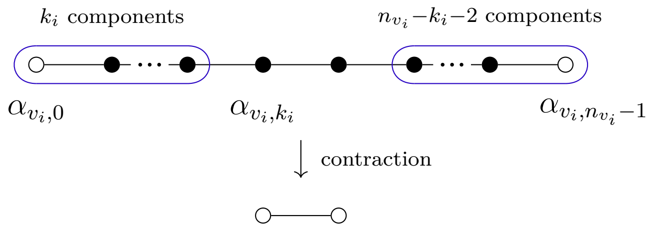


Figura 1 – Blowing-down a fiber of type A_n to a fiber of type A_2

Let G_v be a fiber of type D_n, with $n \geq 3$. Since we assume H intersects the component $\beta_{w_i,0}$, we can similarly blow-down a chain of $n_{w_i} - 2$ components starting with $\beta_{w_i,n_{w_i}\text{-}1}$, reaching a fiber of type A_2 (see Figure 2).

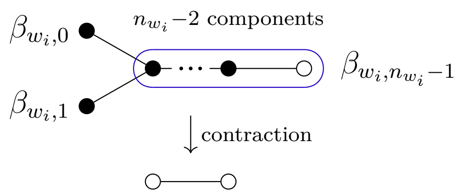


Figura 2 – Blowing-down a fiber of type D_n to a fiber of type A_2

Applying these blow-downs to all reducible fibers of φ, we reach a conic bundle $\varphi_0\colon R_0 \to \mathbb{P}^1$ in which all reducible fibers are of type A_2. Therefore, φ_0 is a standard conic bundle. □

Proposition 2.3.7. *Let $\varphi\colon R \to \mathbb{P}^1$ be a conic bundle over a relatively minimal rational elliptic surface and H a section of φ. The following hold.*

i) There is a free basis of $\mathrm{NS}(\overline{R})$ *given by*

$$\mathcal{B} = \langle G, \alpha_{v_1,1}, \dots, \alpha_{v_1,n_{v_1}\text{-}1}, \dots, \alpha_{v_\delta,1}, \dots, \alpha_{v_\delta,n_{v_\delta}\text{-}1}, \\ H, \beta_{w_1,1}, \dots, \beta_{w_1,n_{w_1}\text{-}1}, \dots, \beta_{w_\varepsilon,1}, \dots, \beta_{w_\varepsilon,n_{w_\varepsilon}\text{-}1}\rangle.$$

ii) The canonical divisor of R is given by

$$K_R = -\,2H + \Big(\sum_{i=1}^{\delta} k_i + H^2 - 2\Big)G + \sum_{i=1}^{\delta}\Big(\sum_{j=1}^{n_{v_i}\text{-}1}(|k_i - j| - k_i)\alpha_{v_i,j}\Big) + \sum_{i=1}^{\varepsilon}\Big(\sum_{j=1}^{n_{w_i}\text{-}1} j\beta_{w_i,j}\Big).$$

Demonstração. Let $\eta\colon R \to R_0$ be the contraction to a standard conic bundle in Proposition 2.3.6. For each reducible fiber with n components, $n-2$ components are contracted. Thus there are $\sum_{i=1}^{\delta}(n_{v_i}-2)+\sum_{i=1}^{\varepsilon}(n_{w_i}-2)$ individual contractions. We can decompose η as

$$\eta = \eta_{v_1,n_{v_1}\text{-}2} \circ \cdots \circ \eta_{v_1,1} \circ \cdots \circ \eta_{v_\delta,1} \circ \eta_{w_1,n_\varepsilon\text{-}2} \circ \cdots \circ \eta_{w_\varepsilon,1},$$

where $\eta_{v_i,j}$ (respectively $\eta_{w_i,j}$) is the j-th contraction of a component of G_{v_i} (respectively G_{w_i}). These maps correspond to blow-ups with the contracted component as the exceptional divisor. Theorem 2.3.4 gives us a free basis of $\mathrm{NS}(\overline{R_0})$ and the canonical divisor K_{R_0}. In what follows, we apply basic properties of blow-ups (see (BEAUVILLE, 1996, Proposition II.3)) to each of the maps $\eta_{v_i,j}$ and $\eta_{w_i,j}$.

i) The image of H by η is a section of the conic bundle $\varphi_0\colon R_0 \to \mathbb{P}^1$. Indeed, by Proposition 2.3.6, the contracted divisors do not intersect H. To simplify notation, we also refer to this section as H. For each fiber G_{v_i}, all of its components are contracted by η, except for α_{v_i,k_i} and α_{v_i,k_i+1}. Similarly, for each fiber G_{w_i}, all components are contracted except for $\beta_{w_i,0}$ and $\beta_{w_i,1}$. We also refer to the image of these components in R_0 by the same notation, and to the fiber class of φ_0 by G. Since H intersects the components α_{v_i,k_i} and $\beta_{w_i,0}$, we know by Theorem 2.3.4 that $\mathrm{NS}(\overline{R_0})$ is generated by the free basis

$$\langle G, H, \alpha_{v_1,k_i+1}, \dots, \alpha_{v_\delta,k_\delta+1}, \beta_{w_1,1}, \dots, \beta_{w_\varepsilon,1}\rangle.$$

By (BEAUVILLE, 1996, Proposition II.3.(iii)), $\mathrm{NS}(\overline{R})$ is generated by by the pullback of the basis of $\mathrm{NS}(\overline{R_0})$ and the exceptional divisors of the blow-ups $\eta_{v_i,j}, \eta_{w_i,j}$. When $k_i = 0$ for all $i = 1, \dots, \delta$, this is equal to $\mathcal{B}$ and we are done. Otherwise, we write $\alpha_{v_i,0}$ in terms of the basis $\mathcal{B}$ as $G-\alpha_{v_i,1}-\ldots-\alpha_{v_i,n_{v_i}\text{-}1}$, so $\mathcal{B}$ generates $\mathrm{NS}(\overline{R})$.

ii) By Theorem 2.3.4, the canonical divisor of R_0 is given by

$$K_{R_0} = -2H + (H^2-2)G + \sum_{i=1}^{\delta} \alpha_{v_i,k_i+1} + \sum_{i=1}^{\varepsilon} \beta_{w_i,1}.$$

We obtain K_R by applying (BEAUVILLE, 1996, Proposition II.3.(iv)) successively for each individual blow-up $\eta_{v_i,j}, \eta_{w_i,j}$. Firstly, notice that since the section H and a general fiber G do not intersect any of the exceptional divisors, their pullbacks by any $\eta_{v_i,j}, \eta_{w_i,j}$ are given by only their strict transforms. Therefore, we focus on calculating the pullbacks on α_{v_i,k_i+1} and $\beta_{w_i,1}$, as well as the exceptional divisors introduced by each blow-up. We can do this fiber by fiber.

We start with a fiber $\varphi_0^{-1}(w_i)$. The map, $\eta_{w_i,1}$ is centered at a point of $\beta_{w_i,1}$, and its exceptional divisor corresponds to $\beta_{w_i,2}$. Thus, we calculate

$$\eta^*_{w_i,1}(\beta_{w_i,1}) + \beta_{w_i,2} = \beta_{w_i,1} + 2\beta_{w_i,2}.$$

Subsequently, the j-th blow-up $\eta_{w_i,j}$ is centered at the component $\beta_{w_i,j}$. Applying this for all j up to $n_{w_i} - 2$, we conclude the part of K_R supported in G_{w_i} is equal to

$$\sum_{j=1}^{n_{w_i}\text{-}1} j\beta_{w_i,j}.$$

For a fiber $\varphi_0^{-1}(v_i)$, the blow-up $\eta_{v_i,1}$ is centered at α_{v_i,k_i+1} with exceptional divisor corresponding to α_{v_i,k_i+2}. We calculate

$$\eta^*_{v_i,1}(\alpha_{v_i,k_i+1}) + \alpha_{v_i,k_i+2} = \alpha_{v_i,k_i+1} + 2\alpha_{v_i,k_i+2}.$$

Subsequently, for $j = 1, \ldots, n_i{-}k_i{-}2$, the blow-up $\eta_{v_i,j}$ is centered at α_{v_i,k_i+j}. Taking their pullbacks successively, we obtain

$$\sum_{j=1}^{n_i\text{-}k_i\text{-}1} j\alpha_{v_i,k_i+j}.$$

The blow-up $\eta_{v_i,n_i\text{-}k_i\text{-}1}$ is centered at α_{v_i,k_i}, which is not a component in the canonical divisor. Therefore we only add the exceptional curve α_{v_i,k_i-1}. For $j = 0, \ldots, k_i - 1$, the blow-up $\eta_{v_i,n_{v_i}\text{-}k_i\text{-}1+j}$ is centered at α_{v_i,k_i-j}, and we conclude that the part of K_R supported in G_{v_i} is equal to

$$\sum_{j=1}^{k_i} j\alpha_{v_i,k_i\text{-}j} + \sum_{j=1}^{n_i\text{-}k_i\text{-}1} j\alpha_{v_i,k_i+j} = \sum_{j=0}^{n_{v_i}\text{-}1} |k_i{-}j|\alpha_{v_i,j}.$$

In order to write K_R in terms of the basis $\mathcal{B}$, we substitute

$$k_i\alpha_{v_i,0} = k_i(G - \alpha_{v_i,1} - \ldots - \alpha_{v_i,n_{v_i}\text{-}1})$$

Thus, we obtain the result. □

2.3.2 Mordell–Weil ranks of rational elliptic surfaces via conic bundles

Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface over and $\varphi\colon R \to \mathbb{P}^1$ a conic bundle over k. In this section, we relate the rank r of $\mathcal{E}(\overline{k}(T))$ to the number δ of fibers of type A_n of φ. Firstly, by Corollary 1.2.4 and Proposition 2.3.2, we know that both r and δ are at most 8. The following proposition shows another way in which these numbers are related.

Proposition 2.3.8. *If $r = 8$, then $\delta = 8$.*

Demonstração. If $r = 8$, then by Corollary 1.2.4 π has no reducible fibers. Consequently, we can use Proposition 1.2.3 to conclude that there are no rational (-2)-curves in X. By Theorem 1.2.11, every reducible fiber of φ is of type A_2, so using Proposition 2.3.2 we obtain $\delta = 8$. □

In what follows, we prove that $\delta \geq r$. We start with a useful definition.

Definition 2.3.9. Let F_v be a fiber of $\pi \colon R \to \mathbb{P}^1$. Then, we define

$$\ell_v := \#\{\Theta \text{ irreducible component of } F_v |\, \Theta \text{ is a fiber component of } \varphi\}.$$

Before the main result, we prove the following Lemmata.

Lemma 2.3.10. *Let F_v be a fiber of $\pi \colon R \to \mathbb{P}^1$, and let m_v be its number of irreducible components. Then, $\ell_v \leq m_v - 1$.*

Demonstração. By Definition 2.3.9, its clear that $\ell_v \leq m_v$. Suppose that for a fiber F_v we have $\ell_v = m_v$. Then, since every component of F_v is a fiber component of φ, we have $F_v \cdot G = 0$. This is not possible, since by adjunction $G \cdot (-K_R) = 2$ and by Proposition 1.2.3, $-K_R = F$, where F is the fiber class of π. □

Lemma 2.3.11. *We can write the sum of ℓ_v over every $v \in \mathbb{P}^1$ as follows:*

$$\sum_{v \in \mathbb{P}^1} \ell_v = \sum_{i=1}^{\delta} (n_{v_i} - 2) + \sum_{i=1}^{\varepsilon} (n_{w_i} - 1)$$

Demonstração. By Proposition 1.2.3, every (-2)-curve in R is a fiber component of π. Therefore,

$$\sum_{v \in \mathbb{P}^1} \ell_v = \#\{\Theta \text{ irreducible component of a fiber of } \varphi |\, \Theta^2 = -2\}.$$

We obtain the result by the classification in Theorem 1.2.11 (see Table 3). □

Lemma 2.3.12. *We can write the difference $\delta - r$ as follows:*

$$\delta - r = \sum_{v \in \mathbb{P}^1} (m_v - 1 - \ell_v).$$

Demonstração. By algebraic manipulation we can write

$$\sum_{v \in \mathbb{P}^1} (n_v - 1) = \delta + \sum_{i=1}^{\delta} (n_{v_i} - 2) + \sum_{i=1}^{\varepsilon} (n_{w_i} - 1).$$

Then, Lemma 2.3.11 yields

$$\sum_{v\in\mathbb{P}^1}(n_v-1)=\delta+\sum_{v\in\mathbb{P}^1}\ell_v.$$

By Proposition 2.3.2 and Corollary 1.2.4, we have

$$\delta+\sum_{v\in\mathbb{P}^1}\ell_v=8=r+\sum_{v\in\mathbb{P}^1}(m_v-1).$$

Rearranging, we obtain the result. □

With this, we are ready to prove the following.

Proposition 2.3.13. *Let R be a rational surface with an elliptic fibration π of Mordell–Weil rank r over $\overline{k}$. Let $\varphi\colon R\to\mathbb{P}^1$ be a conic bundle with δ fibers of type A_n. Then, $\delta\geq r$.*

Demonstração. Applying Lemma 2.3.10, we have

$$\sum_{v\in\mathbb{P}^1}\ell_v\leq\sum_{v\in\mathbb{P}^1}(m_v-1).$$

Thus, Lemma 2.3.12 yields

$$\delta-r=\sum_{v\in\mathbb{P}^1}(m_v-1-\ell_v)\geq 0.$$

□

Remark 2.3.14. Notice that Proposition 2.3.8 is a special case of Proposition 2.3.13.

2.4 Conic bundles via Weierstrass Equations

In this section, we study the elliptic curves $\mathcal{E}$ over $k(T)$ defined by Equation 2.1, with a conic bundle structure induced by Equation 2.2 (see Section 2.1).

$$y^2=a_3(T)x^3+a_2(T)x^2+a_1(T)x+a_0(t), \tag{2.1}$$

$$y^2=A(x)T^2+B(x)T+C(x). \tag{2.2}$$

Let R be the Kodaira–Néron model of $\mathcal{E}$, $\pi\colon R\to\mathbb{P}^1$ the elliptic fibration and $\varphi\colon R\to\mathbb{P}^1$ the conic bundle determined by Equation 2.2. Denote the number of fibers of φ of type A_n by δ and of type D_n by ε. We follow Notation 2.3.1 for the components of the reducible fibers of φ. Recall that each root θ of $\Delta_{\text{conic}}(x)=B(x)^2-4A(x)C(x)$ determines a pair of points $P_\theta,-P_\theta\in\mathcal{E}(\overline{k}(T))$ (see Equation 2.3).

2.4.1 The fiber types of a conic bundle on a rational elliptic surface

Let $\varphi\colon R \to \mathbb{P}^1$ be the conic bundle determined by Equation 2.2. In what follows, we determine the singular fibers of φ and their respective types, as classified in Theorem 1.2.11.

Proposition 2.4.1. *Let $\Delta_{conic}(x) = B(x)^2 - 4A(x)C(x)$, and for every $\theta \in \overline{k}$, let $G_\theta := \varphi^{-1}([\theta{:}1])$. Then, G_θ is singular if and only if $\Delta_{conic}(\theta) = 0$. Moreover, assuming $v_{(x-\theta)}(\Delta_{conic}) = n-1$, the following hold.*

1. *If $A(\theta) \neq 0$ or $C(\theta) \neq 0$, then G_θ is of type A_n.*
2. *If $A(\theta) = C(\theta) = 0$, then G_θ is of type D_n.*

Demonstração. Firstly, assume $A(\theta) \neq 0$. Then, applying the change of coordinates $T = T' - \frac{B(x)}{2A(x)}$, we obtain the following equation.

$$y^2 = A(x)(T')^2 - \frac{\Delta_{\text{conic}}(x)}{4A(x)}.$$

Testing by partial derivatives, the point $y = T' = 0$ is singular in the special fiber at $(x-\theta)$ if and only if $\Delta_{\text{conic}}(\theta) = 0$. If $v_{(x-\theta)}(\Delta_{\text{conic}}(x)) = 1$, then $y = T' = x - \theta = 0$ is regular in X (see (LIU, 2006, Corollary 4.2.12)), and G_θ is of type A_2, composed of the two lines $y = \sqrt{A(\theta)}T'$ and $y = -\sqrt{A(\theta)}T'$. If $v_{(x-\theta)}(\Delta_{\text{conic}}(x)) = n-1$ for $n \geq 3$, then the point $y = T' = x - \theta = 0$ is a singularity of type A_{n-2} (see (REID, , Table 1)). Then, G_θ is of type A_n, with two components coming from the lines $y = \sqrt{A(\theta)}T'$ and $y = -\sqrt{A(\theta)}T'$ and $n-2$ components in the resolution of the singularity at $y = T' = x - \theta = 0$.

Assuming $A(\theta) = 0$ and $C(\theta) \neq 0$, we can apply the change of coordinates $T = 1/u$, $y = y'/u$, arriving at the equation

$$(y')^2 = C(x)u^2 + B(x)u + A(x).$$

Thus, the type of G_θ follows by the previous method.

Finally, assume $A(\theta) = C(\theta) = 0$. Then, $\Delta_{\text{conic}}(x) = B(x)^2$. If $B(\theta) \neq 0$, then the special fiber at $(x-\theta)$ is smooth. Otherwise, the special fiber is a non reduced curve given by $y^2 = 0$. By the classification in Theorem 1.2.11, we know that G_θ is a fiber of type D_n for some $n \geq 3$. By the resolution of the special fiber, $n = v_{(x-\theta)}(\Delta_{\text{conic}}) + 1$. □

Notice that Proposition 2.4.1 does not determine the type for the fiber at infinity. In order do this, we perform the change of coordinates $x \mapsto 1/s$, $y = y'/s^2$, obtaining the following equation.

$$y'^2 = \tilde{A}(s)T^2 + \tilde{B}(s)T + \tilde{C}(s), \tag{2.8}$$

where $\tilde{A}(s) = s^4 A(1/s)$, $\tilde{B}(s) = s^4 B(1/s)$, $\tilde{C}(s) = s^4 C(1/s)$. Define $\tilde{\Delta}(s) := s^8\,\Delta_{\text{conic}}(1/s)$. Since A, B and C have degree at most 3, we know $\tilde{A}(0) = \tilde{B}(0) = \tilde{C}(0) = 0$. An immediate application of Proposition 2.4.1 to Equation (2.8) yields the following.

Proposition 2.4.2. *Assume $v_s(\tilde{\Delta}) = 8 - \deg(\Delta_{conic}) = n - 1$. Then, the fiber at infinity of $\varphi\colon R \to \mathbb{P}^1$ is of type D_n.*

A consequence of Proposition 2.4.2 is that not every conic bundle on a rational elliptic surface can be described by Equations 2.1 and 2.2: see Example 1.2.8 and 1.2.12.

Remark 2.4.3. Notice that we can recover Proposition 2.3.2 by counting the number of components of each reducible fiber in Propositions 2.4.1 and 2.4.2.

2.4.2 The defect of R and the rank of $\mathcal{E}$ over $\overline{k}$

Let $\mathcal{E}$ be an elliptic curve over $k(T)$ defined by Equation 2.1, $\pi\colon R \to \mathbb{P}^1$ its Kodaira–Néron model and $\varphi\colon R \to \mathbb{P}^1$ the conic bundle defined by Equation 2.2. In what follows, we investigate when the rank r of $\mathcal{E}$ over $\overline{k}(T)$ is equal to the number δ of fibers of type A_n of φ. We use the following definition.

Definition 2.4.4. The *defect* of $\mathcal{E}$ is defined as the number

$$\mathrm{Df}(\mathcal{E}) := \delta - r.$$

By Proposition 2.3.13, $\mathrm{Df}(\mathcal{E}) \geq 0$ for any $\mathcal{E}$ defined by Equation 2.1. Since the rank r can be determined through a combination of Tate's Algorithm (see Theorem 1.1.12) and the Shioda–Tate formula (see Corollary 1.1.18), and δ can be determined by Proposition 2.4.1, we can always calculate the defect of $\mathcal{E}$.

Example 2.4.5. Let R be the rational elliptic surface and $\varphi\colon R \to \mathbb{P}^1$ the conic bundle given by the following equation.

$$y^2 = (x^2 - 1)T + x^3 - x + 4.$$

By Tate's algorithm, the elliptic fibration has a single reducible fiber of type I_2^*, thus by the Shioda–Tate formula, $r = 2$. Applying Proposition 2.4.1, φ has two fibers of type A_3. Then, $\delta = 2$ and $\mathrm{Df}(\mathcal{E}) = 0$.

In what follows, we apply the theory of Section 2.3 to $\pi\colon R \to \mathbb{P}^1$ and $\varphi\colon R \to \mathbb{P}^1$. Let $G_{w_1}, \ldots, G_{w_\varepsilon}$ be the fibers of type D_n for some $n \geq 3$. Since by Proposition 2.4.2 the fiber at infinity is of type D_n, we assume without loss of generality that it is G_{w_1}. Let (O) be the zero-section of $\pi\colon R \to \mathbb{P}^1$. By Equation 2.1, (O) is contained in G_{w_1}. Since $(O)^2 = -1$, we conclude that $(O) = \beta_{w_1,n_{w_1}\text{-}1}$.

Proposition 2.4.6. *Let G be the class of a fiber of $\varphi\colon R \to \mathbb{P}^1$. Then, $G \in \mathrm{Triv}(\overline{X})$.*

Demonstração. The fiber class G is represented by G_{w_1}, and we write

$$G_{w_1} = \beta_{w_1,0} + \beta_{w_1,1} + 2\beta_{w_1,2} + \cdots + 2\beta_{w_1,n_{w_1}-2} + 2(O).$$

Since $\beta_{w_1,j}$ is a (-2)-curve for $j = 0, 1, \ldots, n_{w_1} - 2$, they are all fiber components of π. Therefore, $G_{w_1} \in \mathrm{Triv}(\overline{R})$. □

By Proposition 1.2.3, the (-1)-components of singular fibers of φ are sections of π, so they correspond to $\overline{k}(T)$-points of $\mathcal{E}$. We use the following notation according to the type of fiber of φ. For G_{v_i} a fiber of type $A_{n_{v_i}}$, we write $\alpha_{v_i,n_{v_i}-1} = (P_i)$ and $\alpha_{v_i,0} = (P_i')$. For G_{w_i} a fiber of type $D_{n_{w_i}}$, we write $\beta_{w_i,n_{w_i}-1} = (Q_i)$.

Corollary 2.4.7. *In $\mathcal{E}(\overline{k}(T))$, $P_i' = -P_i$ for every $i = 1, \ldots, \delta$, and $[2]Q_i = O$ for every $i = 1, \ldots, \varepsilon$.*

Demonstração. We can write $G_{v_i} \equiv (P_i) + (P_i') \mod \mathrm{Triv}(\overline{R})$, since $\alpha_{v_i,j}$ is a (-2)-curve for $j = 1, \ldots, n_i - 2$. Therefore, by Proposition 2.4.6, $(P_i) + (P_i') \in \mathrm{Triv}(\overline{R})$. Under the isomorphism in Theorem 1.1.17, we deduce $P_i \oplus P_i' = O$. Doing the same for G_{w_i}, we have $2(Q_i) \in \mathrm{Triv}(\overline{R})$, so $[2]Q_i = O$. □

Notice that Corollary 2.4.7 is only true when the conic bundle $\varphi\colon R \to \mathbb{P}^1$ is induced by Equation 2.2. This agrees with the explicit expression for $\overline{k}(T)$-points induced by roots of $\Delta_{\text{conic}}(x)$. Indeed, if G_{v_i} is a fiber of type A_n, then by Proposition 2.4.1, it is equal to G_θ for some $\theta \in \overline{k}$ such that $A(\theta)$ or $C(\theta)$ are non-zero. By Equation 2.3, the points P_θ and $-P_\theta$ are on the line $x = \theta$, so they correspond to the (-1)-components of G_{v_i}. We can assume without loss of generality that P_i corresponds to P_θ. Similarly, if G_{w_i} is a fiber of type D_n, then it is induced by a root θ of Δ_{conic} such that $A(\theta) = C(\theta) = 0$, so by Equation 2.3, P_θ is a point of 2-torsion in $\mathcal{E}(\overline{k}(T))$ corresponding to Q_i.

Definition 2.4.8. Let $S \subset \mathcal{E}(\overline{k}(T))$ be the set $\{P_1, \ldots, P_\delta\}$, M the subgroup of $\mathcal{E}(\overline{k}(T))$ generated by S and L the free $\mathbb{Z}$-module generated over S.

There is a natural surjection $\psi \colon L \to M$ and we have an exact sequence

$$0 \to \ker \psi \to L \xrightarrow{\psi} M \to 0. \tag{2.9}$$

The elements of $\ker \psi$ are equivalent to linear relations between $P_1, \ldots, P_\delta$ in $\mathcal{E}(\overline{k}(T))$.

Theorem 2.4.9. *The rank of* $\ker \psi$ *as a* $\mathbb{Z}$*-module is equal to* $\mathrm{Df}(\mathcal{E})$*.*

Demonstração. Let $z = \mathrm{rank}(\ker \psi)$. By the exact sequence (2.9), we know $z = \delta - \mathrm{rank}\, M$. Since M is a submodule of $\mathcal{E}(\overline{k}(T))$, $\mathrm{rank}\, M \leq r$. Therefore, by Definition 2.4.4,

$$z \geq \delta - r = \mathrm{Df}(\mathcal{E}). \tag{2.10}$$

There are z independent linear relations

$$\begin{aligned} [a_{1,1}]P_1 \quad &\oplus \cdots \oplus \quad [a_{1,\delta}]P_\delta \quad = O, \\ &\vdots \\ [a_{z,1}]P_1 \quad &\oplus \cdots \oplus \quad [a_{z,\delta}]P_\delta \quad = O. \end{aligned}$$

Let $a_i = \sum_{j=1}^{\delta} a_{i,j}$. By Proposition 1.1.22, each linear relation corresponds to an independent vertical divisor $a_{i,1}(P_1) + \ldots + a_{i,\delta}(P_\delta) - a_i(O)$. We use these divisors to write a set of independent divisors of $\mathrm{Triv}(\overline{R})$:

$$\begin{aligned} &F, (O), \\ &\alpha_{v_1,1}, \ldots, \alpha_{v_1,n_{v_1}\text{-}2}, \ldots, \alpha_{v_\delta,1}, \ldots, \alpha_{v_\delta,n_{v_\delta}\text{-}2}, \\ &\beta_{w_1,0}, \ldots, \beta_{w_1,n_{w_1}\text{-}2}, \ldots, \beta_{w_\varepsilon,0}, \ldots, \beta_{w_\varepsilon,n_{w_\varepsilon}\text{-}2}, \\ &a_{1,1}(P_1) + \cdots + a_{1,\delta}(P_\delta) - a_1(O), \\ &\vdots \\ &a_{z,1}(P_1) + \cdots + a_{z,\delta}(P_\delta) - a_z(O). \end{aligned}$$

We can check that the divisors above are linearly independent by writing them in terms of the basis $\mathcal{B}$ in Proposition 2.3.7. By Proposition 1.2.3, $F = -K_F$, and we write K_F in basis $\mathcal{B}$ in Proposition 2.3.7. The components $\alpha_{v_i,j}$ and $\beta_{w_i,j}$ are generators of $\mathcal{B}$ for $j \geq 1$, and we can write

$$\beta_{w_i,0} = G - \beta_{w_i,1} - 2\beta_{w_i,2} - \cdots - 2\beta_{w_i,n_{w_i}\text{-}1}.$$

Finally, we use the correspondences $(O) = \beta_{w_1,n_{w_1}\text{-}1}$ and $(P_i) = \alpha_{v_i,n_{v_i}\text{-}1}$ for the rest of the divisors. Therefore, we have a total of $2 + \sum_{i=1}^{\delta}(n_{v_i} - 2) + \sum_{i=1}^{\varepsilon}(n_{w_i} - 1) + z$ independent divisors in $\mathrm{Triv}(\overline{R})$. Thus, we have

$$2 + \sum_{i=1}^{\delta}(n_{v_i} - 2) + \sum_{i=1}^{\varepsilon}(n_{w_i} - 1) + z \leq \mathrm{rank}(\mathrm{Triv}(\overline{R})) = 2 + \sum_{v \in \mathbb{P}^1}(m_v - 1).$$

By Lemmas 2.3.11 and 2.3.12, we conclude

$$z \leq \mathrm{Df}(\mathcal{E}). \tag{2.11}$$

By the inequalities (2.10) and (2.11), we conclude that $z = \mathrm{Df}(\mathcal{E})$. □

A direct consequence of the previous theorem is the following.

Corollary 2.4.10. *The points $P_1, \ldots, P_\delta$ determined by the conic bundle $\varphi\colon R \to \mathbb{P}^1$ generate a finite index subgroup of $\mathcal{E}(\overline{k}(T))$.*

2.4.3 Bounds on the rank of $\mathcal{E}$ over k

So far, we have worked over the algebraic closure $\overline{k}$ of the field k over which $\pi\colon R \to \mathbb{P}^1$ and $\varphi\colon R \to \mathbb{P}^1$ are defined. In Section 2.4.2, we have studied the relation between r and δ. In order to study the rank r_k of $\mathcal{E}(k(T))$, we define a number δ_k which will play a similar part.

Definition 2.4.11. Let $\mathcal{E}$ be a curve given by Equation 2.2. We define δ_k as

$$\delta_k = \#\left\{[\theta] : \Delta_{\mathrm{conic}}(\theta) = 0, \quad \begin{array}{l} A(\theta) \text{ is a nonzero square in } k(\theta) \text{ or} \\ A(\theta) = 0 \text{ and } C(\theta) \text{ is a nonzero square in } k(\theta) \end{array} \right\},$$

where $[\theta]$ denotes the orbit of θ by the action of $\mathrm{Gal}(\overline{k}/k)$.

Remark 2.4.12. Notice that by Definition 2.4.11, we can rewrite the result of Theorems 2.2.4 and 2.2.5 as follows. If $\mathcal{E}$ is a curve given by an equation of the form 2.5 and $\deg(A) = 0$, then

$$r_{\mathbb{Q}} = \begin{cases} \delta_{\mathbb{Q}} - 1 & \textit{if } A(x) = \mu \in \mathbb{Q}^2 \setminus \{0\}, \\ \delta_{\mathbb{Q}} & \textit{otherwise.} \end{cases}$$

We know $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathrm{NS}(\overline{R})$ preserving the intersection product. In particular, any automorphism $\sigma \in \mathrm{Gal}(\overline{k}/k)$ sends a (-1)-component of a fiber of type A_n to another (-1)-component of a fiber of type A_n. Thus, for any $P_i \in S$, we have $\sigma(P_i) = \pm P_j$. For each $P_i \in S$, let $[P_i]$ be the orbit of P_i by the action of $\mathrm{Gal}(\overline{k}/k)$. The point $\sum_{P\in[P_i]} P$ is invariant under $\mathrm{Gal}(\overline{k}/k)$, so it is a $k(T)$-point of $\mathcal{E}$.

Definition 2.4.13. Let $[\theta_i]$ be an orbit in the set

$$\#\left\{[\theta] : \Delta_{\text{conic}}(\theta) = 0, \quad \begin{array}{l} A(\theta) \text{ is a nonzero square in } k(\theta) \text{ or} \\ A(\theta) = 0 \text{ and } C(\theta) \text{ is a nonzero square in } k(\theta) \end{array}\right\}.$$

Choose an element $\theta_i' \in [\theta_i]$, and let

$$\Sigma_i = \sum_{P\in[P_{\theta_i'}]} P$$

We define the set $S_k := \{\Sigma_1, \ldots, \Sigma_{\delta_k}\}$. Further, we define M_k as the subgroup of $\mathcal{E}(k(T))$ generated by S_k, and L_k as the free $\mathbb{Z}$-module over S_k.

Remark 2.4.14. On Definition 2.4.13, different choices of $\theta_i' \in [\theta_i]$ may lead to different results for Σ_i. Specifically, let $\theta_i', \theta_i'' \in [\theta_i]$ and assume that $\sigma(P_{\theta_i'}) = -P_{\theta_i''}$ for some $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Then, $\sum_{P\in[P_{\theta_i'}]} P = -\sum_{P\in[P_{\theta_i''}]} P$. Notice that any linear combination $[n_1]\Sigma_1 \oplus \ldots \oplus [n_{\delta_k}]\Sigma_{\delta_k}$ induces an equivalent linear combination for any other choice of $\theta_i' \in [\theta_i]$, switching the sign of n_i if necessary.

Before our main theorem, we prove the following lemma on the subgroup M_k of $\mathcal{E}(k(T))$.

Lemma 2.4.15. *Let $M^{\mathcal{G}}$ be the subgroup of M (see Definition 2.4.8) invariant by* $\mathrm{Gal}(\overline{k}/k)$. *Then, $M^{\mathcal{G}} = M_k$.*

Demonstração. By definition, M_k is a subgroup of M invariant by Galois action, so $M_k \subset M^{\mathcal{G}}$. Let $[n_1]P_1 \oplus \ldots \oplus [n_\delta]P_\delta \in M^{\mathcal{G}}$. For each $i = 1, \ldots, \delta$, the point P_i is equal to P_θ (see 2.3) for some $\theta \in \overline{k}$ such that $\Delta_{\text{conic}}(\theta) = 0$, and one of $A(\theta)$ and $C(\theta)$ is nonzero. Assume $A(\theta) \neq 0$ is not a square in $k(\theta)$. Then,

$$P_i = \left(\theta, \sqrt{A(\theta)}\left(T + \tfrac{B(\theta)}{2A(\theta)}\right)\right),$$

and the automorphism $\sqrt{A(\theta)} \mapsto -\sqrt{A(\theta)}$ takes P_i to $-P_i$. Similarly, if $A(\theta) = 0$ and $C(\theta)$ is not a square in $k(\theta)$, $\sqrt{C(\theta)} \mapsto -\sqrt{C(\theta)}$ takes P_i to $-P_i$. Since we assume $[n_1]P_1 \oplus \ldots \oplus [n_\delta]P_\delta$ is invariant under the action of $\mathrm{Gal}(\overline{k}/k)$, we conclude $n_i = 0$.

Now, assume $A(\theta)$ is a nonzero square in $k(\theta)$ or $A(\theta) = 0$ and $C(\theta)$ is a nonzero square in $k(\theta)$. Then, for each $P \in [P_i]$ distinct from P_i, we know $P = \pm P_j$ for some $j \neq i$. If $P_j \in [P_i]$, then $n_i = n_j$, and if $-P_j \in [P_i]$, then $n_i = -n_j$. In both cases, either $(\sum_{P\in[P_i]} P) \in S_k$ or $-(\sum_{P\in[P_i]} P) \in S_k$ (see Remark 2.4.14). Thus, $[n_1]P_1 \oplus \ldots \oplus [n_\delta]P_\delta \in M_k$, and we conclude $M_k = M^{\mathcal{G}}$. □

There is a natural surjection $\psi_k \colon L_k \to M_k$, and we have an exact sequence

$$0 \to \ker \psi_k \to L_k \xrightarrow{\psi_k} M_k \to 0. \tag{2.12}$$

We use this exact sequence to prove our main result.

Theorem 2.4.16. *Let r_k be the rank of $\mathcal{E}(k(T))$. Then, $\delta_k \geq r_k \geq \delta_k - \mathrm{Df}(\mathcal{E})$.*

Demonstração. By the exact sequence 2.12, we have $\delta_k = \operatorname{rank} M_k + \operatorname{rank}(\ker \psi_k)$. Since by Lemma 2.4.15 M_k is a finite index subgroup of $\mathcal{E}(k(T))$, we know $\operatorname{rank}(M_k) = r_k$. On the other hand, each linear relation between points of S_k is a linear relation between points of S, so by Theorem 2.4.9, we have $\operatorname{rank}(\ker \psi_k) \leq \mathrm{Df}(\mathcal{E})$. This proves the result. □

Remark 2.4.17. Notice Theorem 2.4.16 is a generalization of Theorem 2.2.6 to the context of any number field k, and allowing $a_3(T) \neq 1$ in Equation 2.1.

2.5 Computations of the rank

Let $\mathcal{E}$ be a curve given by Equation 2.1, $\pi \colon R \to \mathbb{P}^1$ its Kodaira–Néron model and $\varphi \colon R \to \mathbb{P}^1$ the conic bundle induced by Equation 2.2. In general, Theorem 2.4.16 shows that calculating δ_k determines a range of possible values for r_k, but not r_k itself. In this section, we explore cases in which we can determine r_k explicitly. Specifically, we recover Theorems 2.2.4 and 2.2.5 in the more general context of number fields.

2.5.1 Computation of $Df(\mathcal{E})$

Let G_∞ be the fiber at infinity of the conic bundle $\varphi \colon R \to \mathbb{P}^1$. By Proposition 2.4.2, G_∞ is of type D_n, for $n \geq 3$. If $n = 3$, then there are 2 distinct reducible fibers of π with components in common with G_∞ (see (COSTA, 2024, Theorem 5.2)). If $n \geq 4$, only one reducible fiber has a component in common with G_∞. In what follows, we prove that the type of G_∞ and the Kodaira types of the fibers of π with components in common with G_∞ are sufficient for determining the defect $\mathrm{Df}(\mathcal{E})$.

Theorem 2.5.1. *Let $\mathcal{E}$ be a curve given by Equation 2.1, $\pi \colon R \to \mathbb{P}^1$ its Kodaira–Néron model and $\varphi \colon R \to \mathbb{P}^1$ the induced conic bundle. Let G_∞ be the fiber at infinity of φ of type D_n.*

i) If $n = 3$, then $\mathrm{Df}(\mathcal{E})$ is equal to the number of fibers of type IV or I_m, $m \geq 3$, which have an irreducible component in common with G_∞.

ii) If $n \geq 4$, then $\mathrm{Df}(\mathcal{E}) \leq 1$. Let F_a be the fiber of π with components in common with G_∞. In particular, $\mathrm{Df}(\mathcal{E}) = 1$ in three cases:

1. *if G_∞ is of type D_4 and F_a of type I_m for $m \geq 5$;*
2. *if G_∞ is of type D_5 and F_a of type I_1^*;*
3. *if G_∞ is of type D_6 and F_a of type IV^*.*

For all other configurations of G_∞ and F_a, $\mathrm{Df}(\mathcal{E}) = 0$.

Demonstração. For each fiber F_v of π, recall that we define ℓ_v as the number of components of F_v which are also components of a fiber of φ (see Definition 2.3.9). Then, by Proposition 2.3.12 and Definition 2.4.4,

$$\mathrm{Df}(\mathcal{E}) = \sum_{v \in \mathbb{P}^1} (m_v - 1 - \ell_v). \tag{2.13}$$

Let F_v be a fiber of π which has no components in common with G_∞. If F_v is not reducible, then $m_v = 1$ and $\ell_v = 0$. Assume F_v is reducible and let $\Theta_{v,0}$ be its component intersecting the zero-section (O). We can calculate $\Theta_{0,v} \cdot G_\infty = 2$, so $\Theta_{v,0}$ is a 2-section of φ. On the other hand, the remaining $m_v - 1$ components of F_v do not intersect G_∞, so they are fiber components of φ. Thus, $\ell_v = m_v - 1$ and $m_v - 1 - \ell_v = 0$. Therefore, in order to determine $\mathrm{Df}(\mathcal{E})$, we just need to determine ℓ_v for the fibers F_v which have a component in common with G_∞.

i) Let $n = 3$ and F_a, F_b be the fibers of π which have components in common with G_∞. Let $\Theta_{a,0}, \Theta_{b,0}$ be the components of F_a, F_b intersecting (O), respectively. We can write $G_\infty = \Theta_{a,0} + \Theta_{b,0} + 2(O)$.

If F_a is of type IV or I_m, $m \geq 3$, then $\Theta_{a,0}$ intersects $\Theta_{a,1}$ and $\Theta_{a,m_a\text{-}1}$. Since these components intersect G_∞, they are sections of φ. On the other hand, the remaining $m_a - 3$ components $\Theta_{a,2}, \ldots, \Theta_{a,m_a\text{-}2}$ do not intersect G_∞, so they must be components of a fiber of φ. Thus, $\ell_a = m_a - 2$. If F_a is of one of the remaining Kodaira types, then $\Theta_{a,0}$ only intersects $\Theta_{a,1}$. The other components $\Theta_{a,2}, \ldots, \Theta_{a,m_a\text{-}1}$ do not intersect G_∞, so they are fiber components in φ. Thus, $\ell_a = m_a - 1$. We determine ℓ_b by the same arguments. Substituting every ℓ_v in Equation 2.13, we obtain the result.

ii) Let $n \geq 4$ and F_a be the fiber of π which has a component in common with G_∞. For each n, the Kodaira type of F_a is restricted by the intersection pattern on the (-2)-components of G_∞. We prove the result through the following steps for every possible combination of types of G_∞ and F_a.

1. Determine which components of F_a are also components of G_∞.

2. From the remaining components of F_a, we determine which ones intersect G_∞. These are section (or multi-sections) of φ.

3. The remaining components do not intersect G_∞. Therefore, they are fiber components of φ.

With this, we determine ℓ_a, so we can calculate $\mathrm{Df}(\mathcal{E}) = m_a - 1 - \ell_a$. We do not show these steps explicitly, only the final results in Table 4. The first two columns show the types of G_∞ and F_a. The third column shows the components of F_a which are also fiber components of φ, and the fourth column shows the components which are sections or multi-sections of φ, following the notation in (SCHüTT; SHIODA, 2019, Theorem 5.12). Finally, the fifth column shows the value of $\mathrm{Df}(\mathcal{E})$ for each combination. □

G_∞	F_a	fiber components of φ	(multi)-sections of φ	$\mathrm{Df}(\mathcal{E})$
D_4	I_4	$\Theta_0, \Theta_1, \Theta_3$	Θ_2	0
D_4	$I_{m\geq 5}$	$\Theta_0, \Theta_1, \Theta_3, \ldots, \Theta_{m\text{-}3}, \Theta_{m\text{-}1}$	$\Theta_2, \Theta_{m\text{-}2}$	1
D_5	I_1^*	$\Theta_0, \Theta_1, \Theta_4, \Theta_5$	Θ_2, Θ_3	1
D_5	$I_{m\geq 2}^*$	$\Theta_0, \ldots, \Theta_5, \Theta_7, \ldots, \Theta_{m+4}$	Θ_6	0
D_6	IV^*	$\Theta_0, \Theta_2, \Theta_3, \Theta_4, \Theta_6$	Θ_1, Θ_5	1
D_7	III^*	$\Theta_0, \ldots, \Theta_4, \Theta_6, \Theta_7$	Θ_5	0
D_9	II^*	$\Theta_0, \Theta_2, \ldots, \Theta_8$	Θ_1	0
D_{m+5}	$I_{m\geq 0}^*$	$\Theta_0, \Theta_2, \ldots, \Theta_{m+4}$	Θ_1	0

Tabela 4 – $\mathrm{Df}(\mathcal{E})$ for each configuration of G_∞ and F_a

In particular, Theorem 2.5.1 shows that $\mathrm{Df}(\mathcal{E}) \leq 2$ for every curve $\mathcal{E}$ given by Equation 2.1.

2.5.2 Families of curves with $Df(\mathcal{E}) = 0$

If $\mathcal{E}$ is a curve given by Equation 2.1, then by Theorem 2.4.16, we know $\delta_k \geq r_k \geq \delta_k - \mathrm{Df}(\mathcal{E})$. In general, this is not enough to determine r_k explicitly. The obvious exceptions are the examples in which $\mathrm{Df}(\mathcal{E}) = 0$. In this section, we use Theorem 2.5.1 to find the families of curves for which every member has $\mathrm{Df}(\mathcal{E}) = 0$, and thus $r_k = \delta_k$.

We start by looking at the family of curves given by Equation 2.1 in which $a_3(T)$ is non-constant.

Theorem 2.5.2. *Let $\mathcal{E}$ be a curve given by Equation 2.1, and $\gamma(T) := \Delta_{ell}(T)/a_3(T)^2$. Assume that $\deg(a_3) \geq 1$, $\deg(\gamma) = 8$ and that the resultant $\mathrm{Res}(a_3, \gamma)$ is nonzero. Then, $r_k = \delta_k$.*

Demonstração. Firstly, we put Equation 2.1 in Weierstrass form by applying the coordinate changes $x \mapsto x/a_3(T)$ and $y \mapsto y/a_3(T)$. After clearing the denominator, we have

$$y^2 = x^3 + a_2(T)x^2 + a_1(T)a_3(T)x + a_0(T)a_3(T)^2.$$

The discriminant of $\mathcal{E}$ is given by

$$\begin{aligned}\Delta_{\text{ell}}(T) &= -27a_0^2a_3^4 + 18a_0a_1a_2a_3^3 + a_1^2a_2^2a_3^2 - 4a_0a_2^3a_3^2 - 4a_1^3a_3^3 \\ &= a_3^2(-27a_0^2a_3^2 + 18a_0a_1a_2a_3 + a_1^2a_2^2 - 4a_0a_2^3 - 4a_1^3a_3) \\ &= a_3^2(T)\gamma(T).\end{aligned}$$

Let G_∞ be the fiber at infinity of the conic bundle $\varphi\colon R \to \mathbb{P}^1$. Recall that by Proposition 2.4.2, G_∞ is of type D_n where $n = 9 - \deg(\Delta_{\text{conic}})$.

Assume $a_3(T) = p(T - q)$, where $p \in k^\times$ and $q \in k$. Then, writing $\mathcal{E}$ in the form of Equation 2.2, we have $\deg(A) \leq 2$, $\deg(B) = 3$. Thus, $\deg(\Delta_{\text{conic}}) = 6$ and G_∞ is of type D_3. There are two distinct fibers of $\pi\colon R \to \mathbb{P}^1$ which have components in common with G_∞, namely, the fiber F_q at $T = q$ and the fiber at infinity F_∞. We can use Tate's Algorithm to determine the fiber types of F_q, F_∞. Since $\mathrm{Res}(a_3, \gamma) \neq 0$, q is not a root of $\gamma(T)$, so F_q is of type I_2. Similarly, since $\deg(\gamma) = 8$, F_∞ is of type I_2. By Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 0$.

Assume $a_3(T) = p(T - q_1)(T - q_2)$, where $p \in k^\times$ and $q_i \in \overline{k}$. Calculating Δ_{conic}, we obtain the lead coefficient in x equals $p^2(q_1 - q_2)^2$. If $q_1 \neq q_2$, then $\deg(\Delta_{\text{conic}}) = 6$ and G_∞ is of type D_3. The fibers F_{q_1} and F_{q_2} have components in common with G_∞, and by Tate's Algorithm both are of type I_2. If $q_1 = q_2$, then $\deg(\Delta_{\text{conic}}) \leq 5$ and G_∞ is of type D_n for some $n \geq 4$. By Tate's Algorithm, the fiber F_{q_1} is of type I_4. By Table 4, G_∞ is of type D_4. In both cases, by Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 0$, so by Theorem 2.4.16 $r_k = \delta_k$. □

Notice that the conditions imposed in Theorem 2.5.2 on the coefficients $a_i(T)$ exclude only a Zariski closed set. In this sense, this theorem implies that in the family of curves given by Equation 2.1, a general member $\mathcal{E}$ has $r_k = \delta_k$.

Example 2.5.3. Let $\mathcal{E}$ be given by

$$\begin{aligned}y^2 &= Tx^3 + (T^2 + aT + b)x^2 + (cT^2 + dT + e)x + (fT^2 + gT + h) \qquad (2.14)\\ &= (x^2 + cx + f)T^2 + (x^3 + ax^2 + dx + g)T + (bx^2 + ex + h),\end{aligned}$$

for $a, b, c, d, e, f, g, h \in k$. The condition $\operatorname{Res}(a_3, \gamma) \neq 0$ is equivalent to $\gamma(0) = b^2(e^2 - 4bh) \neq 0$. The condition $\deg(\gamma) = 8$ is equivalent to the coefficient of γ in T^8 being nonzero, that is, $c^2 - 4f \neq 0$. Under these assumptions, $r_k = \delta_k$.

In what follows, we apply Theorem 2.5.1 to recover previous results stated in Section 2.2.

Proposition 2.5.4. *Let $\mathcal{E}$ be a curve given by the following equation, for $A, B, C, D, a, b, c \in k$*

$$\begin{aligned} \mathcal{E}\colon\ y^2 &= x^3T^2 + 2g(x)T - h(x), \text{ where} \\ g(x) &= x^3 + ax^2 + bx + c,\ c \neq 0; \\ h(x) &= (A-1)x^3 + Bx^2 + Cx + D. \end{aligned} \tag{2.15}$$

Assume $\Delta_{conic}(x)$ has 6 distinct nonzero roots which are perfect squares over k. Then, $r_k = 6$.

Demonstração. Firstly, notice that writing $\mathcal{E}$ in the form of Equation 2.2, we have $A(x) = x^3$. Since by assumption every root of Δ_{conic} is a perfect square, we have that $\delta_k = 6$. By Proposition 2.4.2, the fiber G_∞ has type D_3, and by Tate's Algorithm, the Kodaira–Néron model $\pi\colon R \to \mathbb{P}^1$ has two reducible fibers of type I_2. Therefore, Theorem 2.5.1 tells us that $\operatorname{Df}(\mathcal{E}) = 0$. Applying Theorem 2.4.16, we obtain the result. □

This result recovers the calculation of the rank in Theorem 2.2.3. We turn next to curves in which $a_3(T)$ is constant in Equation 2.1 and $A(x) = 0$ in Equation 2.2.

Proposition 2.5.5. *Let $\mathcal{E}$ be a curve given by*

$$\mathcal{E}\colon\ y^2 = B(x)T + C(x), \tag{2.16}$$

where $\deg(B) \leq 2$ and $\deg(C) = 3$. Then, $r_k = \delta_k$.

Demonstração. Let G_∞ be the fiber of φ at infinity. Since $\Delta_{\text{conic}}(x) = B(x)^2$, by Proposition 2.4.2, G_∞ is of type D_n with $n = 9 - 2\deg(B)$.

Assume $\deg(B) = 0$ or $\deg(B) = 1$. Then G_∞ is of type D_9 or D_7, respectively. By Theorem 2.5.1, $\operatorname{Df}(\mathcal{E}) = 0$ irrespective of the type of fiber of π which has components in common with G_∞ (see Table 4).

Assume $\deg(B) = 2$. Then, G_∞ is of type D_5. By Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 1$ if and only if G_∞ has components in common with a fiber of π of type I_1^*. Writing $\mathcal{E}$ in the form of Equation 2.1, we have that $a_3(T)$ is a nonzero constant, and $\deg(a_i) \leq 1$ for $i = 0, 1, 2$. Then, $\deg(\Delta_{\text{ell}}) \leq 3$. By Tate's Algorithm, we deduce π has a fiber of additive reduction with at least 7 components. Thus, G_∞ has components in common with a fiber of type I_m^* for $m \geq 2$, so $\mathrm{Df}(\mathcal{E}) = 0$.

Applying Theorem 2.4.16, we deduce that $r_k = \delta_k$. □

This result generalizes Theorem 2.2.4 to a general number field k.

2.5.3 Families of curves with $Df(\mathcal{E}) > 0$

In this section we study families of curves $\mathcal{E}$ given by Equation 2.1 for which $\mathrm{Df}(\mathcal{E}) > 0$. Then, Theorem 2.4.16 is not enough to determine the rank r_k. We explore cases in which we can use additional information to determine r_k.

Proposition 2.5.6. *Let $\mathcal{E}$ be a curve given by*

$$y^2 = \mu T^2 + B(x)T + C(x), \tag{2.17}$$

where $\mu \in k^\times$, $\deg(B) \leq 2$, $\deg(C) = 3$. Then, $\mathrm{Df}(\mathcal{E}) = 1$.

Demonstração. By Proposition 2.4.2, the conic fiber G_∞ at infinity is of type D_n where $n = 9 - \deg(\Delta_{\text{conic}})$. Let F_a be the fiber of π with components in common with G_∞.

Assume $\deg(B) \leq 1$. Then G_∞ is of type D_6. By Table 4, F_a is of type IV^* or I_1^*. Writing $\mathcal{E}$ in the form of Equation 2.1, we have $\deg(a_0) = 2$, $\deg(a_1) \leq 1$ and $\deg(a_2) = \deg(a_3) = 0$. Therefore, $\deg(\Delta_{\text{ell}}) = 4$, and by Tate's Algorithm, π has fiber of additive reduction at infinity with 7 components. Thus, F_a is of type IV^*, and by Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 1$.

Now, assume $\deg(B) = 2$. Then G_∞ is of type D_5, and F_a is of type I_m^* for some $m \geq 0$. Similarly, writing $\mathcal{E}$ in the form of Equation 2.1, we have $\deg(a_0) = 2$, $\deg(a_1) \leq 1$, $\deg(a_2) = 1$ and $\deg(a_3) = 0$. Therefore $\deg(\Delta_{\text{ell}}) = 5$ and by Tate's Algorithm there is a fiber of additive reduction at infinity with 6 components. Thus F_a is of type I_1^* and by Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 1$. □

In this situation, Theorem 2.4.16 is not enough to determine the rank r_k of $\mathcal{E}$.

Proposition 2.5.7. *Let $\mathcal{E}$ be a curve given by Equation 2.17. Then,*

$$r_k = \begin{cases} \delta_k - 1 & \text{if } \mu \in k^2, \\ \delta_k & \text{if } \mu \in k \setminus k^2. \end{cases}$$

Demonstração. For every $\theta \in \overline{k}$ such that $\Delta_{\text{conic}}(\theta) = 0$, let P_θ be the induced point in $\mathcal{E}(\overline{k}(T))$ (see 2.3). Recall from Definition 2.4.8 that S is the set of points P_θ. By (BATTISTONI; BETTIN; DELAUNAY, 2021, Proposition 12), we know that

$$\sum_{\theta:\Delta(\theta)=0} [n_\theta]P_\theta = O \in \mathcal{E}(\overline{k}(T)), \tag{2.18}$$

where $n_i = v_{(x-\theta)}(\Delta_{\text{conic}})$. Notice that since $\Delta_{\text{conic}}(x) \in k[x]$, we have that $n_{\theta'} = n_\theta$ if $\theta' \in [\theta]$. By Proposition 2.5.6, $\text{Df}(\mathcal{E}) = 1$. Therefore, Theorem 2.4.9 implies that Equation 2.18 is the only linear relation between points of S, up to scalar multiplication.

Assume μ is a square in $k(\theta)$. Then, $\sqrt{\mu} \in k(\theta)$. For any $\sigma \in \text{Gal}(\overline{k}/k)$, we have

$$\sigma(P_\theta) = \begin{cases} \left(\sigma(\theta), \sqrt{\mu}(T + \frac{B(\sigma(\theta))}{2\mu})\right) = P_\theta & \text{if } \sigma \in \text{Gal}(\overline{k}/k(\sqrt{\mu})), \\ \left(\sigma(\theta), -\sqrt{\mu}(T + \frac{B(\sigma(\theta))}{2\mu})\right) = -P_\theta & \text{if } \sigma \notin \text{Gal}(\overline{k}/k(\sqrt{\mu})). \end{cases}$$

If $\mu \in k^2$, then $k(\sqrt{\mu}) = k$. Thus, for each $\Sigma_i \in S_k$ we can write $\Sigma_i = \sum_{\theta' \in [\theta]} P_{\theta'}$ for some $\theta \in \overline{k}$ such that $\Delta_{\text{conic}}(\theta) = 0$. Then, Equation 2.18 determines a linear relation between points of S_k. By Theorem 2.4.16, $r_k = \delta_k - 1$.

If $\mu \in k \setminus k^2$, then for each $\Sigma_i \in S_k$, Σ_i is the sum of $P_{\theta'}$ for half of $\theta' \in [\theta]$, and $-P_\theta$ for the other half. Then, any linear relation between point of S_k induces a linear relation between points of S strictly different from Equation 2.18. Since $\text{Df}(\mathcal{E}) = 1$, this is not possible, so $r_k = \delta_k$. □

Notice that this result generalizes Theorem 2.2.5 to any number field.

In what follows, we go back to curves given by Equation 2.16. We see that if we allow $B(x)$ to be a polynomial of degree 3, then the result of Proposition 2.5.5 no longer holds in general.

Proposition 2.5.8. *Let $\mathcal{E}$ be a curve given by*

$$\mathcal{E}: \quad y^2 = B(x)T + C(x), \tag{2.16}$$

where $B(x) = x^3 + ax^2 + bx + c$ *is a separable polynomial of degree 3 and* $C(x) = \lambda B(x) + \mu$, *for* $\lambda \in k$, $\mu \in k^\times$. *Then,* $\mathrm{Df}(\mathcal{E}) = 1$, *and*

$$r_k = \begin{cases} \delta_k - 1 & \text{if } \mu \in k^2, \\ \delta_k & \text{if } \mu \in k \setminus k^2. \end{cases}$$

Demonstração. Since $\deg(\Delta_{\text{conic}}) = 6$, the fiber G_∞ of φ at infinity is of type D_3. Writing $\mathcal{E}$ in the form of Equation 2.1, we have

$$y^2 = (T+\lambda)x^3 + a(T+\lambda)x^2 + b(T+\lambda)x + c(T+\lambda) + \mu.$$

The fibers of π which have components in common with G_∞ are the fiber $F_{-\lambda}$ at $T = -\lambda$ and F_∞ at infinity. We can calculate that $v_{(T+\lambda)}(\Delta_{\text{ell}}) = 4$, and since $(T+\lambda)$ divides a_2, by Tate's Algorithm, $F_{-\lambda}$ is of type IV.

On the other hand, by calculating the lead coefficient of $\Delta_{\text{ell}}(T)$, we have that $\deg(\Delta_{\text{ell}}) = 6$ if and only if $B(x)$ is separable. Then, by Tate's Algorithm, F_∞ is of type I_0^*. Thus, by Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 1$.

Let $\theta_1, \theta_2, \theta_3$ denote the roots of $B(\theta)$. Then, the point corresponding to θ_i in $\mathcal{E}(\overline{k}(T))$ is $P_i = (\theta_i, \sqrt{\mu})$. By this equation, we have $P_1 \oplus P_2 \oplus P_3 = O$. We prove the formula for r_k by using the same arguments as in Proposition 2.5.7. □

Finally, we return to Example 2.5.3. Modifying the equation, we provide an example of a family of curves with $\mathrm{Df}(\mathcal{E}) = 2$.

Example 2.5.9. Let $\mathcal{E}$ be given by

$$y^2 = Tx^3 + (T^2 + aT + 1)x^2 + (2bT^2 + cT + 2d)x + (b^2T^2 + eT + d^2),$$

for $a, b, c, d, e, f \in k$, and let $\gamma(T) = \Delta_{\text{ell}}(T)/T^2$. By our choice of coefficients, $\deg(\gamma(T)) \leq 7$, and $\gamma(0) = 0$. The fibers F_0 and F_∞ are the fibers of π which have components in common with G_∞, and by Tate's algorithm, both are of type I_m for some $m \geq 3$. By Theorem 2.5.1, $\mathrm{Df}(\mathcal{E}) = 2$.

3 Elliptic fibrations on K3 surfaces with non-symplectic automorphisms of prime order

3.1 Introduction

This chapter is based on the paper (MEIRA, 2024), which was published on Mathematische Nachrichten.

Let k be an algebraically closed field such that $\mathrm{char}(k) = 0$. In this chapter, our main objects of study are K3 surfaces X with a non-symplectic automorphism σ of prime order $p \geq 3$. In particular, we study the elliptic fibrations $\pi\colon X \to \mathbb{P}^1$ on such a K3 surface, and classify them in different ways. We do this through two main approaches. Firstly, we generalize the work of Garbagnati and Salgado in (GARBAGNATI; SALGADO, 2019), (GARBAGNATI; SALGADO, 2020) and (GARBAGNATI; SALGADO, 2024), in which the elliptic fibrations of a K3 surface X are classified with respect to a non-symplectic involution ι according to the action of ι on its fibers. We show that this classification, as well as its main results, can be generalized to non-symplectic automorphisms of higher prime orders.

The Néron–Severi and transcendental lattices of every K3 surface X admitting a non-symplectic automorphism σ of order 3 acting trivially on $\mathrm{NS}(X)$ were classified by Artebani, Sarti and Taki on (ARTEBANI; SARTI, 2008), (TAKI, 2008). We make use of this fact to apply the Kneser–Nishiyama method to determine the ADE-types of every possible elliptic fibration in one of these surfaces. We show that we can classify these elliptic fibrations with respect to the automorphism σ.

3.1.1 Chapter structure

Section 3.2 deals with a few preliminary results. In 3.2.1 we give a brief summary of the distinct ways of classifying elliptic fibrations on K3 surfaces. In 3.2.2 we describe the Kneser–Nishiyama method for determining the ADE-types of elliptic fibrations on K3 surfaces. Finally, in 3.2.3 we present the classification of elliptic fibrations of X with respect to σ a non-symplectic automorphism of prime order.

In Section 3.3, we study elliptic fibrations on K3 surfaces with a non-symplectic automorphism of order 3. In 3.3.1 and 3.3.2 we determine properties of elliptic fibrations $\pi\colon X \to \mathbb{P}^1$ of type 1 and 2 with respect to σ, respectively. In 3.3.3 we relate the different types of elliptic fibrations of X with respect to σ and different classes of linear systems on the minimal resolution $\tilde{R}$ of the quotient X/σ. In 3.3.4, we use the induced linear systems to determine explicitly equations for the generic fibers of elliptic fibrations of type 1 and 2 with respect to σ.

In Section 3.4, we apply the Kneser–Nishiyama method to all K3 surfaces X with Picard number at least 12 admitting a non-symplectic automorphism σ of order 3, under the assumption that the automorphism acts trivially on its Néron–Severi group. Table 61 shows every fibration in these surfaces, their respective ADE-types and their Mordell–Weil groups. Sections 3.4.1, 3.4.2 and 3.4.3 show the explicit calculations involved in the application of the Kneser–Nishiyama method. In 3.4.4, we classify the fibrations in Table 6 with respect to the automorphism σ.

In Section 3.5 we illustrate our method described in Section 3.3 by applying it to the X_3 surface. This surface was first studied by Shioda and Inose in (INOSE; SHIODA, 1977) and Vinberg in (VINBERG, 1983), and in (NISHIYAMA, 1996), Nishiyama presented a $\mathcal{J}_2$-classification of Jacobian elliptic fibrations of X_3. We exhibit Weierstrass equations for each fibration in $\mathcal{J}_2(X_3)$ (see Theorem 3.5.6, 3.5.5).

Finally, Section 3.6 deals with generalizing the results of Section 3.3 to non-symplectic automorphisms of prime order $p > 3$. In 3.6.1, we determine the necessary ramified fibers so that the base change of a rational elliptic surface by a Galois cover of degree p becomes a K3 surface. In 3.6.2, we show that every elliptic fibration on a K3 surace X with a non-symplectic automorphism of prime order $p > 3$ acting trivially on $\mathrm{NS}(X)$ comes from such a base change. We use this fact to deduce the Kodaira types of the possible reducible fibers on these elliptic fibrations, and to determine explicit equations for their generic fibers.

3.2 Preliminaries

3.2.1 Classification of elliptic fibrations on K3 surfaces

Let X be a K3 surface. By Proposition 1.3.13, every embedding of the hyperbolic lattice U into $\mathrm{NS}(X)$ induces an elliptic fibration on X. Consequently, when $\mathrm{NS}(X)$ allows for multiple distinct embeddings of U, X admits multiple elliptic fibrations. As such, it is useful to define ways in which two elliptic fibrations π and π' of X are equivalent. The classifications presented here are studied in depth in (BRAUN; KIMURA; WATARI, 2013).

Definition 3.2.1. Let π, π' be elliptic fibrations on X with respective zero-sections s, s'. Then

i) π and π' are $\mathcal{J}_0$-equivalent if there is $g \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\pi' = g \circ \pi$ and $s' = s \circ g$.

ii) π and π' are $\mathcal{J}_1$-equivalent if there is $g \in \mathrm{Aut}(\mathbb{P}^1)$ and $\sigma \in \mathrm{Aut}(X)$ such that $\pi' = g \circ \pi \circ \sigma$ and $s' = \sigma \circ s \circ g$.

The sets of elliptic fibrations on X modulo $\mathcal{J}_0$ and $\mathcal{J}_1$-equivalence are denoted as $\mathcal{J}_0(X)$ and $\mathcal{J}_1(X)$, respectively.

Notice that if π and π' are $\mathcal{J}_0$ equivalent, then they are $\mathcal{J}_1$ equivalent by taking id_X as the automorphism. The $\mathcal{J}_1$ classification is particularly important because of the following theorem.

Theorem 3.2.2. *On any given K3 surface X, there are finitely many elliptic fibrations up to $\mathcal{J}_1$-equivalence.*

Demonstração. See (STERK, 1985, Proposition 2.6, Corollary 2.7). □

If π and π' are $\mathcal{J}_1$-equivalent, then the frame lattices W_π and $W_{\pi'}$ are isomorphic. This motivates a third type of classification.

Definition 3.2.3. Let π, π' be elliptic fibrations of X. We say that they are $\mathcal{J}_2$-equivalent if $W_\pi \cong W_{\pi'}$. The set of elliptic fibrations of X modulo $\mathcal{J}_2$-equivalence is denoted by $\mathcal{J}_2(X)$.

This definition allows us to translate the classification problem to pure lattice theory.

Definition 3.2.4. Let L be an even lattice. The *discriminant group* of L is defined as $G_L := L^\vee/L$, where $L^\vee$ is the dual lattice of L. The *discriminant form* of L is a map $q_L \colon G_L \to \mathbb{Q}/2\mathbb{Z}$, given by $x \mapsto \langle x, x \rangle \mod 2\mathbb{Z}$. The pair (G_L, q_L) is called the *discriminant lattice* of L.

Let $\mathcal{J}_2'(X)$ be the set of all even lattices, modulo isometries, with signature $(0, \rho(X) - 2)$ and discriminant lattice isomorphic to $(G_{\mathrm{NS}(X)}, q_{\mathrm{NS}(X)})$. Then, $\mathcal{J}_2(X)$ is a subset of $\mathcal{J}_2'(X)$.

Proposition 3.2.5. *Let X be a K3 surface with $\rho(X) \geq 12$. Then, $\mathcal{J}_2'(X) = \mathcal{J}_2(X)$.*

Demonstração. See (SCHüTT; SHIODA, 2010, Lemma 12.21). □

3.2.2 The Kneser–Nishiyama method

Based on lattice theoretic techniques developed by Kneser in (KNESER, 1956), Nishiyama developed a method of obtaining $\mathcal{J}_2'(X)$ for K3 surfaces X with known $\mathrm{NS}(X)$ and T_X (see (NISHIYAMA, 1996)). In particular, by Proposition 3.2.5, this method determines $\mathcal{J}_2(X)$ when $\rho(X) \geq 12$. This is known in the literature as the Kneser–Nishiyama method. In what follows we present a brief overview of it. See (BERTIN et al., 2015, Section 4.1) or (BRAUN; KIMURA; WATARI, 2013, Section 4.1) for a similar overview.

Definition 3.2.6. A *Niemeier Lattice* is an even, unimodular, negative definite lattice of rank 24.

Theorem 3.2.7. *Niemeier lattices are uniquely defined by their root types up to isometry, of which there are only 24 possibilities.*

Demonstração. See (NIEMEIER, 1973, Theorem 8.5). □

Let X be a K3 surface with transcendental lattice T_X.

Theorem 3.2.8. *Let T_0 be a lattice of root type such that $\operatorname{rank} T_0 = \operatorname{rank} T_X + 4$, $G_{T_0} = G_{T_X}$ and $q_{T_0} = q_{T_X}$. Then, every $W \in \mathcal{J}_2'(X)$ can be written as $\varphi(T_0)^{\perp L}$, for $\varphi\colon T_0 \to L$ a primitive embedding into a Niemeier lattice L. Furthermore, if $\rho(X) \geq 12$, then there is an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ such that $W = W_\pi$, and the following holds.*

i) Let $M = \varphi(T_0)^{\perp L_{\mathrm{root}}}$. Then the ADE-type of π is M_{root} and isomorphic to W_{root}.

ii) The rank of $\mathrm{MW}(\pi)$ is given by $\operatorname{rank} M - \operatorname{rank} M_{\mathrm{root}}$.

iii) The torsion part of $\mathrm{MW}(\pi)$ is isomorphic to $\overline{M_{\mathrm{root}}}/M_{\mathrm{root}}$, where $\overline{M_{\mathrm{root}}}$ is the primitive closure of M_{root}.

Demonstração. See (NISHIYAMA, 1996, Section 6.1, Section 6.2). □

The Kneser–Nishiyama method consists of the application of the previous result to obtain $\mathcal{J}_2'(X)$. We describe it in the following steps.

1. Find a suitable lattice T_0 of root type.

2. For each Niemeier lattice L, determine every possible primitive embedding $\varphi\colon T_0 \to L_{\text{root}}$, up to actions of the Weyl group $\mathrm{W}(L)$ (see (BOURBAKI, 1968, Chapter 6, Definition 1)).

3. For each embedding φ, compute the orthogonal lattices $M := \varphi(T_0)^{\perp L_{\text{root}}}$ and the root type $M_{\text{root}} = W_{\text{root}}$.

4. Compute the rank of the Mordell–Weil group, given by $\operatorname{rank} M - \operatorname{rank} M_{\text{root}}$, and the torsion, given by $\overline{M_{\text{root}}}/M_{\text{root}}$.

3.2.3 Classification of fibrations with respect to a non-symplectic automorphism

In (GARBAGNATI; SALGADO, 2019), Garbagnati and Salgado define a classification of elliptic fibrations on a K3 surface in relation to a non-symplectic involution ι. This definition generalizes nicely to any non-symplectic automorphism σ of prime order. In this section, we reproduce this definition and show some of its main properties. Firstly, we fix some notation.

Notation 3.2.9. Let S be a surface and σ an automorphism of S. We say that a point $p \in S$ is *fixed* by σ if $\sigma(p) = p$. Let $C \subset S$ be a curve. We say that C is *fixed* by σ if every point $p \in C$ is fixed by σ, i.e. if $\sigma|_C = \mathrm{id}_C$, and C is *preserved* by σ if C is not fixed by σ but $\sigma(C) = C$.

Definition 3.2.10. Let (X, σ) denote a pair consisting of X a K3 surface, and σ a fixed non-symplectic automorphism of X of prime order p. We classify an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ with respect to σ as follows.

1) π is of *type 1* if every $F_v = \pi^{-1}(v)$ is preserved by σ.

2) π is of *type 2* if σ fixes the fiber class F in $\mathrm{NS}(X)$, but σ is not of type 1 (i.e. there exist distinct $v, v' \in \mathbb{P}^1$ such that $\sigma(F_v) = F_{v'}$).

3) π is of *type 3* if σ does not fix the class of the fiber F in $\mathrm{NS}(X)$.

Remark 3.2.11. Assume σ acts trivially on $\mathrm{NS}(X)$. Then, in particular, σ fixes the fiber class, so (X, σ) does not admit elliptic fibrations of type 3.

The following two propositions are adapted from (GARBAGNATI; SALGADO, 2020, Proposition 2.5, Theorem 2.6), considering non-symplectic automorphisms of any prime order. We present the proofs for completeness.

Proposition 3.2.12. *Suppose π is an elliptic fibration of type 1 on (X, σ), and σ acts trivially in* $\mathrm{NS}(X)$. *Then every section of π is fixed by σ. Consequently,* $\operatorname{rank} \mathrm{MW}(\pi) = 0$.

Demonstração. Let $C \subset X$ be a smooth rational curve. Since σ acts trivially on $\mathrm{NS}(X)$, and rational curves are unique in their class on K3 surfaces, $\sigma(C) = C$. Then, if Σ is a section of π, since σ acts trivially on the base of the fibration, Σ must be fixed by σ. The number of curves fixed by σ is finite, so rank $MW(\pi) = 0$. □

Proposition 3.2.13. *Assume (X, σ) admits an elliptic fibration of type 2. Then, every curve fixed by σ has genus $g \leq 1$.*

Demonstração. Let $\pi \colon X \to \mathbb{P}^1$ be a type 2 elliptic fibration of (X, σ), and $C \subset X$ a curve fixed by σ. Suppose C is a multi-section of π. Then, C intersects any fiber with positive multiplicity, and the intersection points must be fixed. However, since π is of type 2, there are distinct fibers F_v and F_u such that $\sigma(F_v) = F_u$, so F_v does not have fixed points. Therefore, C must be a fiber component of π, so $g(C) \leq 1$. □

3.3 K3 surfaces with non-symplectic automorphisms of order 3

In Section 3.2.3, we have seen how to classify elliptic fibrations of a K3 surface in relation to a non-symplectic automorphism of prime order p. The case of $p = 2$ was studied extensively by Garbagnati and Salgado in (GARBAGNATI; SALGADO, 2019), (GARBAGNATI; SALGADO, 2020) and (GARBAGNATI; SALGADO, 2024). In what follows, we deal with the case $p = 3$. The choice of focusing in this order comes from Proposition 3.3.1, in which we see that fibrations of type 1 do not occur with respect to automorphisms of higher prime order.

3.3.1 Fibrations of type 1

Let $\pi \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 1 on (X, σ). Since the automorphism σ preserves every fiber of π, we can consider the restriction of its action to said fibers. This allows us to deduce properties of both σ and the singular fibers of π.

Proposition 3.3.1. *Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of prime order p. If (X, σ) admits an elliptic fibration $\pi \colon X \to \mathbb{P}^1$ of type 1, then $p = 2$ or 3. Furthermore, if $p = 3$, the singular fibers of π must be of type I_0^*, II, IV, II^* or IV^*.*

Remark 3.3.2. This list of possible fibers for an elliptic fibrations of type 1 with respect to a non-symplectic automorphism of order 3 is given in (OHASHI; TAKI, 1989, Proposition 3.5.(4)), with the added assumption that σ fixes a curve of genus $g \geq 2$. We present the proof for completeness.

Demonstração. Assume (X, σ) admits an elliptic fibration $\pi \colon X \to \mathbb{P}^1$ of type 1. Then, σ acts as an automorphism of order p on each fiber. In particular, for a smooth fiber F_v, σ acts as an automorphism of a genus 1 curve. By (SILVERMAN, 1986, Chapter X, Proposition 5.1), σ corresponds with (P, α), where P is a point of F_v and α an automorphism of F_v as an elliptic curve. In particular, $\sigma = \tau_P \circ \alpha$, where τ_P is the translation by P map. Thus, assuming $\alpha = \mathrm{id}$, we obtain that σ acts on X as the translation by a section of π. If this was true, then σ would be a symplectic automorphism. Since by hypothesis σ is non-symplectic, we can assume $\alpha \neq \mathrm{id}$ for every smooth F_v. Then, $\sigma^p = \mathrm{id}$ corresponds to $(\sum_{i=0}^{p-1} \alpha^i(P), \alpha^p)$, so the order of α is p. By (SILVERMAN, 1986, Chapter III, Theorem 10.1), the only admissible automorphism groups of elliptic curves are $\mathbb{Z}/2$, $\mathbb{Z}/4$ and $\mathbb{Z}/6$, so p can only be 2 or 3.

If $p = 3$, then there is a $\mathbb{Z}/6$ action on each smooth fiber of π, and their short Weierstrass form must be $y^2 = x^3 + B$. Consequently, the J-function of π is constant and equal to zero. The only types of singular fibers with $J(F) = 0$ are I_0^*, II, IV, II^* and IV^* (see (MIRANDA, 1989, Table IV.3.1)). □

Remark 3.3.3. If $p = 2$ and the quotient X/σ is a relatively minimal rational elliptic surface, the singular fibers of a fibration of type 1 were classified in (GARBAGNATI; SALGADO, 2019, Theorem 5.3).

In what follows, we often work under the assumption that σ acts trivially on the Néron–Severi group of X. When this is the case, we obtain the following improvement on the result of Proposition 3.3.1.

Proposition 3.3.4. *Let X be a K3 surface and σ a non-symplectic automorphism of order 3 acting trivially on* $\mathrm{NS}(X)$*. If $\pi \colon X \to \mathbb{P}^1$ is an elliptic fibration of type 1 on (X, σ), then π does not admit fibers of type I_0^*.*

Demonstração. By the action of σ, we can write the equation for the generic fiber of π as $y^2 = x^3 + B(t)$. In this equation, we can see the automorphism σ explicitly as $(x, y, t) \mapsto (\zeta_3 x, y, t)$, where $\zeta_3 = \frac{-1+i\sqrt{3}}{2}$ is the cubic root of unity. Assume π has a singular fiber F_v of type I_0^*. By Tate's algorithm, after a suitable change of coordinates, we can write this equation as $y^2 = x^3 + t^3 f(t)$, where $f(t) \neq 0$ and

F_v is the resolution of the singular fiber at $t = 0$. After blowing-up the singularity at the origin, we obtain the equation $Y^2 = t(X^3 + f(t))$, where $X = x/t$ and $Y = y/t$. This equation yields a surface with 3 A_1 singularities at $(\sqrt[3]{-f(t)}, 0, 0)$, $(\zeta_3\sqrt[3]{-f(t)}, 0, 0)$ and $(\zeta_3^2\sqrt[3]{-f(t)}, 0, 0)$. The action of σ is lifted to the blow-up by $(X, Y, t) \mapsto (\zeta_3 X, Y, t)$, so the A_1 singularities are permuted. Thus, σ permutes the 3 components of F_v coming from the resolution of the A_1 singularities, and σ acts nontrivially on $\mathrm{NS}(X)$. □

When σ acts trivially on $\mathrm{NS}(X)$, sufficient conditions for the existence of an elliptic fibration of type 1 on (X, σ) were given in (ARTEBANI; SARTI, 2008).

Proposition 3.3.5. *If σ acts trivially in $\mathrm{NS}(X)$ and fixes at least 2 curves, then (X, σ) admits an elliptic fibration of type 1.*

Demonstração. See (ARTEBANI; SARTI, 2008, Proposition 4.2). □

3.3.2 Fibrations of type 2

We begin this section by describing a natural way to exhibit explicitly a pair (X, σ) with X a K3 surface and σ a non-symplectic automorphism admitting an elliptic fibration of type 2. We do this by starting from a rational elliptic surface $\pi\colon R \to \mathbb{P}^1$ and applying the base change by a Galois covering $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$. By resolving singularities and contracting (-1)-curves of the resulting surface $R \times_{\mathbb{P}^1} \mathbb{P}^1$, we obtain a surface X with a relatively minimal elliptic fibration $\pi_X\colon X \to \mathbb{P}^1$. The Galois morphism on $\mathbb{P}^1$ lifts to an automorphism of X of order equal to the degree of $\tau_{\mathbb{P}^1}$. Assume that X is a K3 surface. Since the quotient by σ is birational to R, by Theorem 1.3.12 σ is non-symplectic. Furthermore, fibers of π_X above points of $\mathbb{P}^1$ outside the branch locus of $\tau_{\mathbb{P}^1}$ are permuted, so π_X is of type 2 on (X, σ).

Proposition 3.3.6. *Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface, $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ a cubic Galois covering ramified at $a, b \in \mathbb{P}^1$ and F_a, F_b the fibers of π above a and b. The surface X obtained taking the base change of π by $\tau_{\mathbb{P}^1}$ is a K3 if and only if one of F_a, F_b is of type I_n^* or IV, while the other is of type I_n, II or III.*

Demonstração. By the canonical divisor formula for elliptic surfaces (see (SCHüTT; SHIODA, 2019, Theorem 5.28)), Noether's Formula (see (BEAUVILLE, 1996, I.14)) and Serre duality ((BEAUVILLE, 1996, Theorem I.11)), the surface X in an elliptic fibration with basis $\mathbb{P}^1$ is a K3 surface if and only if its Euler number $e(X)$ (see (SCHüTT; SHIODA, 2019, Section 4.7)) is equal to 24. By (SCHüTT; SHIODA, 2019, Theorem 5.31), $e(X) = \sum_{v\in\mathbb{P}^1} e(F_v^X)$, where $F_v^X = \pi_X^{-1}(v)$. Let u, v be points in $\mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}(v) = u$. Then, the Euler number of F_v^X is known in terms of the Kodaira type of F_u and the ramification index $r(v|u)$ (see (MIRANDA, 1989, Table VI.4.1)). Knowing that $e(R) = \sum_{u\in\mathbb{P}^1} e(F_u) = 12$, we obtain the result by checking for which Kodaira types of F_a and F_b we obtain $e(X) = 24$. □

Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of order 3. In what follows, we aim to use Proposition 3.3.6 to describe the configuration of fibers on an elliptic fibration of type 2 on (X, σ). As a first step, we study the properties of the quotient X/σ.

By Theorem 1.3.12, we know X/σ is rational, but in general, it is not a rational elliptic surface. Let $x \in X$ be a fixed point of σ. By Proposition 1.3.10 the local action of σ around a fixed point $x \in X$ can be linearized as

$$A = \begin{pmatrix} \zeta_3^i & 0 \\ 0 & \zeta_3^j \end{pmatrix},$$

where $\zeta_3 = \frac{-1+i\sqrt{3}}{2}$ is the cubic root of unity. Since σ is non-symplectic, we deduce that $i = 1$ and $j = 0$, or $i = j = 2$. In the former case, since $j = 0$, x is part of a fixed curve. In the latter case, x is an isolated fixed point. Notice that by Theorem 1.3.9, both cases are admissible. Assume x is an isolated fixed point, and let $\tau\colon X \to X/\sigma$ be the quotient map. Then, by the action of A, we can infer that $\tau(x)$ is a singularity of type $\frac{1}{3}(1,1)$ (see (REID, 2003)). In order to circumvent this, we can first blow-up the isolated fixed points of σ.

Proposition 3.3.7. *Let $\eta_X\colon \tilde{X} \to X$ be the blow-up of the isolated fixed points of σ. Then, the following statements are true.*

i) Every elliptic fibration $\pi\colon X \to \mathbb{P}^1$ lifts to an elliptic fibration $\tilde{\pi}\colon \tilde{X} \to \mathbb{P}^1$, and both fibrations are isomorphic on an open set of $\mathbb{P}^1$.

ii) We can lift σ to an automorphism $\tilde{\sigma}$ of $\tilde{X}$ which fixes the exceptional curves of η_X.

iii) Let $\tilde{R} := \tilde{X}/\tilde{\sigma}$, and $\tilde{\tau}\colon \tilde{X} \to \tilde{R}$ be the quotient map. Then, $\tilde{R}$ is isomorphic to the minimal resolution φ of X/σ, and the following diagram commutes.

$$\begin{array}{ccc} \tilde{R} & \longleftarrow & \tilde{X} \\ \downarrow & & \downarrow \\ X/\sigma & \longleftarrow & X \end{array}$$

Demonstração. i) Let $[F]$ be the class of a fiber and Σ_0 the zero-section of π in $\mathrm{NS}(X)$, and write $\tilde{F} := \eta_X^*(F)$ and $\tilde{\Sigma}_0 := \eta_X^*(\Sigma_0)$. Since a general smooth fiber of π does not intersect the isolated fixed points of σ, the pencil $|\tilde{F}|$ induces an elliptic fibration on $\tilde{X}$, and we can choose $\tilde{\Sigma}_0$ as the zero-section.

ii) The surface $\tilde{X}$ can be described locally around an exceptional divisor by coordinates (z_1, z_2, t) with $z_2 = tz_1$. We define $\tilde{\sigma}$ by the map $(z_1, z_2, t) \mapsto (\zeta_3^2 z_1, \zeta_3^2 z_2, t)$. Then, $\tilde{\sigma}$ agrees with σ outside of $z_1 = z_2 = 0$, and fixes the exceptional curve.

iii) Let $Z = \{P_1, ..., P_n\}$ be the set of isolated fixed points of σ, and $\tilde{Z} = \{E_1, ..., E_n\}$ the set of corresponding exceptional curves on $\tilde{X}$. The open subsets $X \setminus Z$ and $\tilde{X} \setminus \tilde{Z}$ are isomorphic, and σ and $\tilde{\sigma}$ agree under this identification. By taking the respective quotients, we obtain that the open sets $Y \setminus \varphi^{-1}(\tau(Z))$ and $\tilde{R} \setminus \tilde{\tau}(\tilde{Z})$ are isomorphic. For any isolated fixed point P_i, we know that $E_i^2 = -1$ and it is fixed by $\tilde{\sigma}$. Then $C_i = \tilde{\tau}(E_i)$ is rational and its self intersection is $C_i^2 = C_i \cdot \tilde{\tau}_*(E_i) = E_i \cdot \tilde{\tau}^*(C_i) = 3E_i^2 = -3$. By (REID, 2003, Example 3.1) the minimal resolution of a singularity of type $\frac{1}{3}(1,1)$ is a rational curve with self-intersection -3, so Y and $\tilde{R}$ are isomorphic as claimed. □

Remark 3.3.8. For a K3 surface with a non-symplectic automorphism of higher prime orders, Proposition 3.3.7 does not necessarily hold. For example, assume σ is a non-symplectic automorphism of order 5 and p an isolated fixed point with local action given by $(z_1, z_2) \mapsto (\zeta_5^2 z_1, \zeta_5^4 z_2)$. Extending the action of σ to the blow-up of p, we obtain $(z_1, z_2, t) \mapsto (\zeta_5^2 z_1, \zeta_5^4 z_2, \zeta_5^2 t)$. In this case, the exceptional divisor is not fixed by $\tilde{\sigma}$. Consequently, $\tilde{\sigma}$ still has isolated fixed point and the quotient $\tilde{X}/\tilde{\sigma}$ is singular.

Proposition 3.3.9. *Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ), and assume σ preserves the zero-section. Then, π induces an elliptic fibration $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$, so $\tilde{R}$ is a rational elliptic surface.*

Demonstração. By Proposition 3.3.7, π induces an elliptic fibration $\tilde{\pi}$ on $\tilde{X}$. Let $\tilde{F}$ be the fiber class and $\tilde{\Sigma}_0$ the zero-section of $\tilde{\pi}$. Let $\tilde{\tau}\colon \tilde{X} \to \tilde{R}$ be the quotient by $\tilde{\sigma}$, and denote $D = \tilde{\tau}(\tilde{F})$ and $C = \tilde{\tau}(\tilde{\Sigma}_0)$. We claim that the pencil $|D|$ induces an elliptic fibration on $\tilde{R}$, and we can choose C as the zero-section.

Since π is of type 2, every curve in the fixed locus of σ is a fiber component. For all but finitely many choices of $v_1 \in \mathbb{P}^1$, there are three distinct smooth fibers $\tilde{F}_{v_1}$, $\tilde{F}_{v_2}$ and $\tilde{F}_{v_3}$ in an orbit of $\tilde{\sigma}$. Consequently, for a generic choice of D_v in the pencil $|D|$, the map $\tilde{\tau}$ defines a cubic covering of D_v by 3 disjoint smooth genus 1 curves. By the Riemann–Hurwitz Theorem, D_v must also be smooth of genus 1, and $|D|$ a genus 1 pencil. Furthermore, $D^2 = \tilde{\tau}_*\tilde{F} \cdot D_v = \tilde{F} \cdot \tilde{\tau}^*(D_v) = \tilde{F} \cdot (\tilde{F}_{v_1} + \tilde{F}_{v_2} + \tilde{F}_{v_3}) = 0$, so $|D|$ induces an elliptic fibration $\pi_{\tilde{R}} \colon \tilde{R} \to \mathbb{P}^1$. Since $\tilde{\Sigma}_0$ is preserved by $\tilde{\sigma}$, we have $\tilde{\tau}^*(C) = \tilde{\Sigma}_0$, and we can calculate the intersection product $D \cdot C = \tilde{\tau}_*(\tilde{F}_1) \cdot C = \tilde{F}_1 \cdot \tilde{\Sigma}_0 = 1$. Thus, C is a section, and we conclude that $\pi_{\tilde{R}}$ is an elliptic fibration with C as the zero-section. □

Let R be the relatively minimal model of $\tilde{R}$ with respect to the elliptic fibration $\pi_{\tilde{R}}$. There is a blow-down $\eta_R \colon \tilde{R} \to R$ and R is endowed with a relatively minimal elliptic fibration $\pi_R \colon R \to \mathbb{P}^1$ such that $\pi_{\tilde{R}} = \eta_R \circ \pi_R$.

Proposition 3.3.10. *Let $\pi_X \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ), and assume σ preserves zero-section. Then, there is a map $\tau_{\mathbb{P}^1} \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\pi \colon X \to \mathbb{P}^1$ is the base change of the rational elliptic surface $\pi_R \colon R \to \mathbb{P}^1$ by $\tau_{\mathbb{P}^1}$, and σ is the induced automorphism.*

Demonstração. Let $s_0 \colon \mathbb{P}^1 \to X$ be the zero-section of π and $s_0(\mathbb{P}^1) = \Sigma_0$. Notice that since Σ_0 is preserved by σ, we can define an automorphism of $\mathbb{P}^1$ as $\sigma_{\mathbb{P}^1} = \pi \circ \sigma \circ s_0$. Since s_0 is a section, we have $s_0 \circ \pi|_{\Sigma_0} = \mathrm{id}_{\Sigma_0}$, thus $\sigma_{\mathbb{P}^1}^3 = \pi \circ \sigma^3 \circ s_0 = \mathrm{id}_{\mathbb{P}^1}$. Furthermore, by the definition of fibrations of type 2, $\sigma_{\mathbb{P}^1}$ acts nontrivially on $\mathbb{P}^1$, so it has order 3. Let $\tau_{\mathbb{P}^1} \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the quotient map by $\sigma_{\mathbb{P}^1}$.

Let $U \subset \mathbb{P}^1$ be an open set such that $\pi_{\tilde{R}}^{-1}(v)$ is a smooth fiber for every $v \in U$. Then, $\pi_{\tilde{R}}^{-1}(U)$ is isomorphic to $\pi_R^{-1}(U)$, and the base change of π_R by $\tau_{\mathbb{P}^1}$ must be birational to $\pi_{\tilde{X}} \colon \tilde{X} \to \mathbb{P}^1$. After resolving singularities and contracting (-1)-curves on the fibers, by the uniqueness of the relatively minimal model of elliptic surfaces, we obtain the fibration $\pi_X \colon X \to \mathbb{P}^1$. Furthermore, since the automorphism induced by this base change agrees with σ on the dense open set $\pi_R^{-1}(U)$, they must agree everywhere. □

Both Proposition 3.3.9 and 3.3.10 have the hypothesis that σ preserves the zero-section. Indeed, the Galois morphism obtained by taking the base change of a rational elliptic surface will always fix this class. In the following proposition we show that this condition is necessary.

Proposition 3.3.11. *Let $\pi \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 in (X, σ) such that none of its sections is preserved by σ. Then the induced map $\pi_{\tilde{R}} \colon \tilde{R} \to \mathbb{P}^1$ in Proposition 3.3.9 is a fibration in genus 1 curves without section.*

Demonstração. Let $\tilde{\pi}$ be the induced elliptic fibration on $\tilde{X}$, $\tilde{F}$ its fiber class and $\widetilde{\Sigma_1}, \widetilde{\Sigma_2}, \widetilde{\Sigma_3}$ an orbit of its sections by the action of $\tilde{\sigma}$. Let $\tilde{\tau} \colon \tilde{X} \to \tilde{R}$ be the quotient by $\tilde{\sigma}$, and denote $D = \tilde{\tau}(\tilde{F})$ and $C = \tilde{\tau}(\tilde{\Sigma_1}) = \tilde{\tau}(\tilde{\Sigma_2}) = \tilde{\tau}(\tilde{\Sigma_3})$. We can calculate the intersection $D \cdot C = \tilde{\tau}_*(\tilde{F}) \cdot C = \tilde{F} \cdot \tilde{\tau}^*(C) = \tilde{F} \cdot (\tilde{\Sigma_1} + \tilde{\Sigma_2} + \tilde{\Sigma_3}) = 3$. Thus, C is a 3-section of $\pi_{\tilde{R}} \colon \tilde{R} \to \mathbb{P}^1$. □

Indeed, the conditions for Proposition 3.3.11 can happen. Let π is a fibration of type 2 on (X, σ) such that σ preserves its zero-section and π has a 3-torsion section P. Then the translation by P determines a symplectic automorphism $\sigma' \colon X \to X$. Thus we can construct a non-symplectic automorphism $\sigma'' = \sigma \circ \sigma'$ such that none of the sections of π are preserved by σ''.

We can use Proposition 3.3.6 to prove the following.

Proposition 3.3.12. *Let $\pi_X \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ), and assume σ preserves the zero-section. Then, σ preserves two fibers F_a^X and F_b^X, and every other fiber is in an orbit $F_{v_1}^X, F_{v_2}^X, F_{v_3}^X$ of σ. Furthermore, up to permuting F_a^X and F_b^X we have the following.*

i) F_a^X is of type I_0 or I_n^* for $n = 0, 3, 6, 9, 12$.

ii) F_b^X is of type I_0^*, III^* or I_m for $m = 0, 3, 6, 9, 12, 15, 18$.

iii) $F_{v_1}^X, F_{v_2}^X$ and $F_{v_3}^X$ have the same type, which can be II, III, IV, IV^*, I_n^* for $n = 0, 1$ or I_m for $m = 0, 1, \ldots, 6$.

iv) There are no fibers of type II^*, I_n^* for $n = 2, 4, 5, 7, 8, 10, 11, 13$ or I_n for $n = 7, 8, 10, 11, 13, 14, 16, 17, 19$.

Demonstração. By Proposition 3.3.10, π is the base change of a rational elliptic surface $\pi_R \colon R \to \mathbb{P}^1$ by a Galois covering $\tau_{\mathbb{P}^1} \colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 3 ramified over $a, b \in \mathbb{P}^1$. By Proposition 3.3.6, we know F_a is of type IV or I_n^* and F_b is of type II, III or I_m. Since R is rational, we know by the Shioda–Tate formula (Corollary 1.1.18) that fibers of π_R have at most 9 components, if F_u is of type I_m or I_n^*, we know $m \leq 9$ and $n \leq 4$.

Let F_a^X, F_b^X be the fibers of π above F_a, F_b respectively. Then, by (MIRANDA, 1989, Table VI.4.1), we know F_a^X is of type I_0 or I_n^* for $n = 0, 3, 6, 9, 12$, and F_b^X is of type I_0^*, III^* or I_m, for $m = 0, 3, 6, 9, 12, 15, 18$. This proves (i) and (ii).

Let F_u be a fiber of π_R for $u \neq a, b$. Since u is not ramified by $\tau_{\mathbb{P}^1}$, there are three distinct points $v_1, v_2, v_3 \in \mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}(v_i) = u$, and $F_{v_i}^X$ has the same type as F_u. By (PERSSON, 1990), we know that any of the fiber types listed in (iii) are possible for F_u.

It remains to prove that the types listed in (iv) are not possible for F_u. Firstly, notice that F_a has at least 3 components, so by Shioda–Tate F_u has at most 7 components. Therefore, it remains to check for I_2^* and I_7. Since both have 7 components, we know F_a is of type IV and F_b is irreducible. However, these configurations are impossible (see (MIRANDA, 1990, Table 2.1, No. 22 and 27)). □

Corollary 3.3.13. *Assume that σ acts trivially on $\mathrm{NS}(X)$. Then, every fiber other than F_a^X and F_b^X is irreducible.*

Demonstração. Assume F_v^X is not irreducible, for $v \neq a, b$. Then, σ takes the components of F_v^X to the components of another fiber. Since fiber components are independent in $\mathrm{NS}(X)$, this constitutes a non-trivial action. □

3.3.3 Classification by induced linear systems

Let X be a K3 surface with a non-symplectic involution ι. In work by Garbagnati and Salgado, the elliptic fibrations of X are directly related to linear systems on the quotient X/ι, which is shown to be a rational elliptic surface with the assumption that ι fixes curves of genus at most 1 (see (GARBAGNATI; SALGADO, 2019), (GARBAGNATI; SALGADO, 2020)). Our goal is to study the linear systems induced by elliptic fibrations on the resolution $\tilde{R}$ of the quotient X/σ (see 3.3.7). In order to do this, we work with the following assumption through the rest of this section.

Assumption 3.3.14. Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of order 3. We assume that X admits an elliptic fibration $\pi_X : X \to \mathbb{P}^1$ of type 2 with respect to σ such that σ preserves the zero-section.

Let (X, σ, π_X) be a K3 surface with Assumption 3.3.14. Let π be an elliptic fibration on X (possibly distinct from π_X). Then π induces a pencil of curves Λ on $\tilde{R}$ by pulling back $|F|$ by η_X, and then applying the pushforward by $\tilde{\tau}$. The following theorem describes the relation between the type of an elliptic fibrations in 3.2.10 and which kind of pencil it induces on $\tilde{R}$ (see Definitions 1.2.9 and 1.2.13)

Theorem 3.3.15. *The induced pencil Λ is determined by the type of π.*

i) π is of type 1 if and only if Λ is a conic bundle class of $\tilde{R}$.

ii) π is of type 2 if and only if Λ is a splitting genus 1 pencil of $\tilde{R}$.

iii) π is of type 3 if and only if Λ is a non-complete linear system.

The proof of this theorem is a direct adaptation of (GARBAGNATI; SALGADO, 2019, Theorem 4.2), in which the automorphism σ is a non-symplectic involution. The biggest change is the necessity of pulling back the linear system to $\tilde{X}$ before taking the quotient.

Demonstração. Let $[\tilde{F}]$ be the class of a fiber of the elliptic fibration induced in $\tilde{X}$ by π and $\tilde{F}_v$ the pullback of F_v.

Suppose that π is of type 1. Then the action of σ on each F_v lifts to an action of $\tilde{\sigma}$ on $\tilde{F}_v$. The pencil Λ is given by the system of curves $\{D_v\}_{v\in\mathbb{P}^1}$, where $D_v = \tilde{\tau}(\tilde{F}_v) = \tilde{F}_v/\tilde{\sigma}$. Since σ has a finite number of isolated fixed points, for all but finitely many $v \in \mathbb{P}^1$, F_v is smooth and $D_v = F_v/\sigma$. Applying the Riemann–Hurwitz Theorem to the quotient map $F_v \to D_v$, we know that $g(D_v) = 0$ if and only if the map ramifies in two distinct points with index 3, and $g(D_v) = 1$ if and only if it is unramified. If we assume $g(D_v) = 1$, then σ acts as the translation of a torsion point of F_v as an elliptic curve, fixing its period. Furthermore, since this is true for all but finitely many $v \in \mathbb{P}^1$ and σ acts as the identity on the base of π, then σ must preserve the period of X. That is not possible due to the assumption that σ is non-symplectic, so $g(D_v) = 0$. We can calculate the self intersection as $D_v^2 = D_v \cdot \tilde{\tau}(\tilde{F}_v) = \tilde{\tau}^*(D_v) \cdot \tilde{F}_v = 3\tilde{F}_v \cdot \tilde{F}_v = 0$. By the adjunction formula, $D_v \cdot K_{\tilde{R}} = -2$. We conclude that $\Lambda = |D_v|$ is a generalized conic bundle of R (with respect to $\eta_R \colon \tilde{R} \to R$).

Suppose π is of type 2. By Proposition 3.3.9, we know that Λ consists of the system of fibers $\{D_v\}_{v\in\mathbb{P}^1}$ in an elliptic fibration. Consequently, $D_v^2 = 0$ and $g(D_v) = 1$, and by the adjunction formula, $D_v \cdot K_{\tilde{R}} = 0$. Therefore, Λ is a splitting genus 1 pencil.

Suppose π is of type 3. Then $\sigma([F]) = [F']$ and $\sigma([F']) = [F'']$, for $[F], [F'], [F'']$ three distinct classes on $\mathrm{NS}(X)$, each respectively inducing distinct elliptic fibrations π, π', π''. Pulling back these classes by η_X, we obtain $[\tilde{F}], [\tilde{F}'], [\tilde{F}'']$ distinct classes in $\mathrm{NS}(\tilde{X})$. Since they are supported on smooth curves of $\tilde{X}$, the intersection products $\tilde{F}\tilde{F}'$, $\tilde{F}\tilde{F}''$ and $\tilde{F}'\tilde{F}''$ are all greater than 0. Let $\tilde{F}\tilde{F}' + \tilde{F}\tilde{F}'' + \tilde{F}'\tilde{F}'' = m > 0$, then $(\tilde{F} + \tilde{F}' + \tilde{F}'')^2 = 2m$. Since $\tilde{F}^2 = \tilde{F}'^2 = \tilde{F}''^2 = 0$, the linear system $|\tilde{F} + \tilde{F}' + \tilde{F}''|$ is base point free. In particular, there is a smooth curve C_X of genus $m+1$ whose class is $[\tilde{F} + \tilde{F}' + \tilde{F}'']$. As a consequence, $|C_X| = |\tilde{F} + \tilde{F}' + \tilde{F}''|$ is an $m+1$ dimensional linear system with smooth general elements (see (SAINT-DONAT, 1974, Proposition 2.6)). On the other hand, the family of curves $\tilde{F}_v + \tilde{F}'_v + \tilde{F}''_v$, given by $\eta_X^{-1}(\pi^{-1}(v) + \pi'^{-1}(v) + \pi''^{-1}(v))$ for each $v \in \mathbb{P}^1$, has dimension 1 and reducible general element. Taking $D_v = \tilde{\tau}(\tilde{F}_v) = \tilde{\tau}(\tilde{F}'_v) = \tilde{\tau}(\tilde{F}''_v)$, we conclude $\Lambda = \{D_v\}_{v\in\mathbb{P}^1}$ is a non-complete sub-linear system of $|\tilde{\tau}(C_X)|$. □

3.3.4 Equations for elliptic fibrations of type 1 and 2

Let (X, σ, π_X) denote a K3 surface, a non symplectic automorphism of order 3 and an elliptic fibration of type 2 following Assumption 3.3.14. By Proposition 3.3.10, π_X is the base change of a rational elliptic fibration $\pi_R\colon R \to \mathbb{P}^1$ by a map $\tau_{\mathbb{P}^1}: \mathbb{P}^1 \to \mathbb{P}^1$. The fibration π_R is constructed as a resolution η of a rational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by $[x{:}y{:}z] \mapsto [\mathcal{F}(x,y,z) : \mathcal{G}(x,y,z)]$, and after a change of coordinates, we can assume $\tau_{\mathbb{P}^1}$ is given by $[s{:}t] \mapsto [s^3{:}t^3]$. Thus, the generic fiber of π_X can be written as

$$\pi_X\colon \mathcal{F}(x,y,z) + t^3\mathcal{G}(x,y,z) = 0.$$

Now, let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration distinct from π_X, and Λ the induced linear system in $\tilde{R}$. Through the contractions $\eta_{\tilde{R}}\colon \tilde{R} \to R$ and $\eta\colon R \to \mathbb{P}^2$, Λ induces a pencil of curves Γ in $\mathbb{P}^2$. In this section, we show how to use Γ to deduce an equation for the generic fiber of π, when π is of type 1 or 2 in relation to σ.

Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration of type 1 on (X, σ). Then, Λ is a pencil of rational curves in $\tilde{R}$.

Proposition 3.3.16. *Let the restriction of $\pi_R\colon \tilde{R} \to \mathbb{P}^1$ to D_v be given by the map*

$$\begin{aligned} f_v\colon D_v &\to \mathbb{P}^1 \\ P &\mapsto [x_v(P) : y_v(P)]. \end{aligned}$$

Then, we can write the generic fiber of $\pi\colon X \to \mathbb{P}^1$ as

$$\pi\colon s^3 x_v = t^3 y_v.$$

Demonstração. Let F_v be a smooth fiber of π such that F_v is isomorphic to $\tilde{F}_v = \eta_X^{-1}(F_v)$. Notice that the following diagram commutes

$$\begin{array}{ccc} D_v & \longleftarrow & F_v \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longleftarrow & \mathbb{P}^1. \end{array}$$

By the universal property of fiber products, there is a 1-1 map $F_v \to D_v \times_{\mathbb{P}^1} \mathbb{P}^1$. Since F_v is smooth, this must be an isomorphism. Then, we can write F_v in coordinates

$(P,[s:t])$, where P is a point in D_v and $[s:t]$ on $\mathbb{P}^1$ such that $f_v(P) = \tau_{\mathbb{P}^1}([s{:}t])$, that is, $[x_v(P){:}y_v(P)] = [s^3{:}t^3]$. This gives rise to the proposed equation for the generic fiber. □

Now, let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X,σ) distinct from π_X. By Proposition 3.3.10, we know that π is the base change of a rational elliptic surface by a cubic Galois covering $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$.

Proposition 3.3.17. *Let F_a, F_b be the fibers above the ramification points of $\tau_{\mathbb{P}^1}$, and $\mathcal{C}_a$, $\mathcal{C}_b$ the induced curves in Γ. Then, we can write the generic fiber of π by the following equation*

$$\pi\colon \mathcal{C}_a(x,y,z) + t^3\mathcal{C}_b(x,y,z) = 0.$$

Demonstração. By Theorem 3.3.15, we know that Λ is a genus 1 pencil inducing an elliptic fibration $\pi'\colon \tilde{R} \to \mathbb{P}^1$. Then, Γ must be a pencil of genus 1 curves in $\mathbb{P}^2$ generated by $\mathcal{C}_a$ and $\mathcal{C}_b$. For all but finitely many $t \in \mathbb{P}^1$, the fiber $(\pi')^{-1}(t)$ is isomorphic to $\mathcal{C}_a(x,y,z) + t\mathcal{C}_b(x,y,z) = 0$. By a change of coordinates, we can suppose that $\tau_{\mathbb{P}^1}$ is given by the map $t \mapsto t^3$. Thus, applying the base change by $\tau_{\mathbb{P}^1}$, we obtain the wanted equation for the generic fiber of π. □

Remark 3.3.18. In order to use this proposition, we need to know what are the fibers of $\pi\colon X \to \mathbb{P}^1$ above the ramification locus of $\tau_{\mathbb{P}^1}$. We can deduce this from the ADE-type of π. By Proposition 3.3.12, every reducible fiber which is the only one of its Kodaira type must be ramified by the base change, otherwise it would have 3 copies. For instance, if the ADE-type of π is $D_7 \oplus E_7$, then the only reducible fibers are of type I_3^* and III^*, and both need to be ramified by $\tau_{\mathbb{P}^1}$.

3.3.5 Conic bundles inducing elliptic fibrations

Let $\pi\colon R \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface and $\varphi\colon R \to \mathbb{P}^1$ a conic bundle. Assume X is a K3 surface obtained by taking the base change of $\pi\colon R \to \mathbb{P}^1$ by a Galois cover $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2, and let ι be the induced non-symplectic involution. Then, the conic bundle φ induces an elliptic fibration on X of type 1 with respect to ι (see (GARBAGNATI; SALGADO, 2019, Theorem 5.3)).

In this section, we study the same phenomenon for base changes of degree 3. We work under the assumption that the base change of $\pi\colon R \to \mathbb{P}^1$ by a Galois cover $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ produces a K3 surface X (that is, by Proposition 3.3.6 one of the ramified fibers is of type I_n^* or IV and the other is of type I_n, II or III). This induces a non-symplectic automorphism σ of order 3 on X. By Theorem 3.3.15, we know that every elliptic fibration $\pi_X\colon X \to \mathbb{P}^1$ of type 1 on (X, σ) comes from a conic bundle on R. We ask the inverse question: when does a conic bundle $\varphi\colon X \to \mathbb{P}^1$ determine an elliptic fibration of type 1 on (X, σ)?

We can see that in general this does not happen. Indeed, let D_v be a smooth fiber of φ and let $f_v\colon D_v \to \mathbb{P}^1$ be the restriction of π_R to D_v. Then, the fiber B_v of π_φ corresponding to D_v is isomorphic to the fiber product of $f_v\colon D_v \to \mathbb{P}^1$ by $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$. We can calculate the ramification of the map $B_v \to D_v$ in order to find the genus $g(B_v)$. By Definition 1.2.9, we know that $D_v \cdot F = 2$, where F is the class of fibers of π_R. Then, f_v has degree 2, and ramifies at two distinct points $c, d \in \mathbb{P}^1$. Assume that a, b, c, d are all distinct. Then, there are distinct points $a_1, a_2, b_1, b_2 \in D_v$ such that $f_v(a_i) = a$, $f_v(b_i) = b$ for $i = 1, 2$. Furthermore, let a_0, b_0 be points of $\mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}(a_0) = a$ and $\tau_{\mathbb{P}^1}(b_0) = b$. Then, a_1, a_2, b_1, b_2, are the ramification point of $B_v \to D_v$, each having a single point in its pre-image, given by $(a_1, a_0), (a_2, a_0), (b_1, b_0), (b_2, b_0)$ respectively. Using the Riemann–Hurwitz Theorem, we can calculate that $g(D_v) = 2$.

Proposition 3.3.19. *Let $\varphi\colon R \to \mathbb{P}^1$ be a conic bundle in R, and let a, b be the points in $\mathbb{P}^1$ where $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ ramifies. Then, φ induces an elliptic fibration in X if and only if the map $f_v\colon C_v \to \mathbb{P}^1$ given by the restriction of π_R to the fiber $D_v := \varphi^{-1}(v)$ ramifies in either a or b for every $v \in \mathbb{P}^1$.*

Demonstração. Suppose $f_v\colon D_v \to \mathbb{P}^1$ ramifies in a for every $v \in \mathbb{P}^1$, and assume that the other ramification point is distinct from b. Then, the map $D_v \times_{\mathbb{P}^1} \mathbb{P}^1 =: B_v \to D_v$ ramifies in a', b_1, b_2, where $f_v(a') = a$ and $f_v(b_1) = f_v(b_2) = b$. Applying the Riemann–Hurwitz Theorem, we have $g(B_v) = 1$. On the other hand, if φ induces an elliptic fibration on X, then for all but finitely many $v \in \mathbb{P}^1$ it is true that $g(B_v) = 1$, and by the Riemann–Hurwitz Theorem the map $B_v \to D_v$ must ramify in 3 points. Let c_v, d_v be the ramification points of f_v. If c_v, d_v are distinct from a, b, then $B_v \to D_v$ would ramify in 4 distinct points, and $g(C'_v) = 2$. Then we can assume without loss of generality that $c_v = a$ for every $v \in \mathbb{P}^1$. □

Example 3.3.20. Let $\pi\colon R \to \mathbb{P}^1$ be the rational elliptic surface induced by the pencil of cubics $\Lambda = s\mathcal{F} + t\mathcal{G}$ in $\mathbb{P}^2$, where $\mathcal{F} = y^2z - x^3 + xz^2 - 4z^3$ and $\mathcal{G} = (x+z)(x-z)z$. The pencil $\Lambda_P = \alpha x - \beta z$ describes the lines of $\mathbb{P}^2$ through the point $P = [0{:}1{:}0]$. Since P is a base point of Λ, this induces a conic bundle $\varphi\colon R \to \mathbb{P}^1$. We want to show that φ defines an elliptic fibration on the base change of X through the map $\tau_{\mathbb{P}^1}([s{:}t]) = [s^3{:}t^3]$.

Firstly, notice that $\tau_{\mathbb{P}^1}$ ramifies in the point $[0{:}1], [1{:}0]$. For every $v = [\alpha{:}\beta] \in \mathbb{P}^1$, the line $\alpha x = \beta z$ in Λ_P is the image of the following map

$$\begin{aligned} \rho_v \colon \mathbb{P}^1 &\longrightarrow \mathbb{P}^2 \\ [u_1{:}u_2] &\longmapsto [\beta u_1{:}u_2{:}\alpha u_1]. \end{aligned}$$

We can evaluate the map $f_v \colon C_v \to \mathbb{P}^1$ as the resolution of the composition of ρ_v with the cubic map $[x{:}y{:}z] \mapsto [\mathcal{F}(x,y,z){:}\mathcal{G}(x,y,z)]$, thus obtaining

$$\begin{aligned} f_v \colon \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ [u_1{:}u_2] &\longmapsto [\alpha u_2^2 + (\alpha^2\beta - \beta^3 - 4\alpha^3)u_1^2 {:} (\alpha\beta^2 - \alpha^3)u_1^2]. \end{aligned}$$

Then, for each $v = [\alpha{:}\beta]$ the ramification points of f_v are $[1{:}0]$ and $[\alpha^2 - \beta^3 - 4\alpha^3 {:} ab^2 - a^3]$. Since $[1{:}0]$ is a ramification point of $\tau_{\mathbb{P}^1}$ and of f_v for every $v \in \mathbb{P}^1$, φ induces an elliptic fibration on the base change X of R by $\tau_{\mathbb{P}^1}$.

3.4 A $\mathcal{J}_2$-classification of K3 surfaces with non-symplectic automorphisms of order 3

In this section, our goal is to provide a $\mathcal{J}_2$-classification to K3 surfaces X with a non-symplectic automorphism σ of order 3 acting nontrivially on $\mathrm{NS}(X)$. We further assume that the Picard number of X is at least 12. Then, by Proposition 3.2.5, the Kneser–Nishiyama method provides the full $\mathcal{J}_2$-classification (see Section 3.2.2). By work of Artebani and Sarti, $\mathrm{NS}(X)$ must be equal to one of 10 possible lattices, already assuming $\rho(X) \geq 12$ (see (ARTEBANI; SARTI, 2008, Proposition 3.2)). For each possibility, Table 5 shows explicitly the Néron–Severi and transcendental lattices, as well as the number n of isolated fixed points of σ, the number m of fixed curves of σ, and g the greatest genus amongst the fixed curves. Note that by (ARTEBANI; SARTI, 2008, Theorem 3.3), for each line in Table 5 there exists a K3 surface X with $\rho(X) \geq 12$, $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of order 3, and the corresponding lattices $\mathrm{NS}(X), T_X$.

Remark 3.4.1. The classification in Table 5 is expanded to higher prime orders in (ARTEBANI; SARTI; TAKI, 2011, Tables 2–7). Let X be a K3 surface such that $\rho(X) \geq 12$ and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of prime order $p > 3$ acting trivially on $\mathrm{NS}(X)$. There are exactly 4 possibilities for the lattices $\mathrm{NS}(X), T_X$. We do not apply the Kneser–Nishiyama method in these cases because there is no suitable T_0 of root type (see Theorem 3.2.8).

No.	NS(X)	T_X	n	m	g
1	$U \oplus A_2^{\oplus 5}$	$U \oplus U(3) \oplus A_2^{\oplus 3}$	5	2	0
2	$U \oplus E_6 \oplus A_2^{\oplus 2}$	$U \oplus U(3) \oplus E_6$	5	3	1
3	$U \oplus E_8 \oplus A_2$	$U^{\oplus 2} \oplus E_6$	5	4	2
4	$U \oplus E_6 \oplus A_2^{\oplus 3}$	$U \oplus U(3) \oplus A_2^{\oplus 2}$	6	3	0
5	$U \oplus E_6^{\oplus 2}$	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	6	4	1
6	$U \oplus E_6^{\oplus 2} \oplus A_2$	$U \oplus U(3) \oplus A_2$	7	4	0
7	$U \oplus E_6 \oplus E_8$	$U^{\oplus 2} \oplus A_2$	7	5	1
8	$U \oplus E_6 \oplus E_8 \oplus A_2$	$U \oplus U(3)$	8	5	0
9	$U \oplus E_8^{\oplus 2}$	$U^{\oplus 2}$	8	6	1
10	$U \oplus E_8^{\oplus 2} \oplus A_2$	$A_2(-1)$	9	6	0

Tabela 5 – Genera and lattices for each pair (n, m)

Theorem 3.4.2. *Let X be a K3 surface with $\rho(X) \geq 12$ and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism acting trivially on* NS(X)*. Then, Table 6 describes the $\mathcal{J}_2$-classification of elliptic fibrations of X. Each fibration is given with its respective ADE-type T, and Mordell–Weil group* MW(π)*.*

Demonstração. The proof of this theorem consists of a direct application of the Kneser–Nishiyama method (see (NISHIYAMA, 1996)). An overview of this method is described in the end of Section 3.2.2, and the explicit computations for the required cases are presented in Sections 3.4.1, 3.4.2 and 3.4.3. □

Tabela 6 – All possible elliptic fibrations on K3 surfaces with non-symplectic automorphisms of order 3

1. $\mathrm{NS}(X) = U \oplus A_2^{\oplus 5}$, $T_X = U \oplus U(3) \oplus A_2^{\oplus 3}$, $T_0 = E_6 \oplus A_2^{\oplus 4}$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
1.1	$E_8^{\oplus 3}$	$E_6 \subset E_8$	$(A_2^{\oplus 2})^{\oplus 2} \subset E_8^{\oplus 2}$		$A_2^{\oplus 5}$	$A_2^{\oplus 5}$	0
1.2	$E_8 \oplus D_{16}$	$E_6 \subset E_8$	$A_2^{\oplus 4} \subset D_{16}$		$A_2 \oplus (A_2^{\oplus 4})^{\perp D_{16}}$	$D_4 \oplus A_2$	$\mathbb{Z}^4$
1.3	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2 \subset E_7$	$A_2^{\oplus 3} \subset D_{10}$	$(-6) \oplus A_5 \oplus (A_2^{\oplus 3})^{\perp D_{10}}$	A_5	$\mathbb{Z}^5$
1.4	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2^{\oplus 2} \subset E_7$	$A_2^{\oplus 2} \subset D_{10}$	$(-6)^{\oplus 2} \oplus A_2 \oplus (A_2^{\oplus 2})^{\perp D_{10}}$	$D_4 \oplus A_2$	$\mathbb{Z}^4$
1.5	$E_7 \oplus A_{17}$	$E_6 \subset E_7$	$A_2^{\oplus 4} \subset A_{17}$		$(-6) \oplus (A_2^{\oplus 4})^{\perp A_{17}}$	A_5	$\mathbb{Z}^5$
1.6	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$(A_2^{\oplus 2})^{\oplus 2} \subset E_6^{\oplus 2}$		$E_6 \oplus A_2^{\oplus 2}$	$E_6 \oplus A_2^{\oplus 2}$	0
1.7	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset E_6$	$(A_2)^{\oplus 2} \subset E_6^{\oplus 2}$	$A_2^{\oplus 5}$	$A_2^{\oplus 5}$	0
1.8	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 4} \subset A_{11}$		$D_7 \oplus (A_2^{\oplus 4})^{\perp A_{11}}$	D_7	$\mathbb{Z}^3$
1.9	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2 \subset D_7$	$A_2^{\oplus 3} \subset A_{11}$	$A_2^{\perp D_7} \oplus (A_2^{\oplus 3})^{\perp A_{11}}$	$D_4 \oplus A_2$	$\mathbb{Z}^4$
1.10	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset D_7$	$A_2^{\oplus 2} \oplus A_{11}$	$(A_2^{\oplus 2})^{\perp D_7} \oplus (A_2^{\oplus 2})^{\perp A_{11}}$	A_5	$\mathbb{Z}^5$

2. $\mathrm{NS}(X) = U \oplus E_6 \oplus A_2^{\oplus 2}$, $T_X = U \oplus U(3) \oplus E_6$, $T_0 = E_6^{\oplus 2} \oplus A_2$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
2.1	$E_8^{\oplus 3}$	$E_6^{\oplus 2} \subset E_8^{\oplus 2}$	$A_2 \subset E_8$		$E_6 \oplus A_2^{\oplus 2}$	$E_6 \oplus A_2^{\oplus 2}$	0
2.2	$E_7^{\oplus 2} \oplus D_{10}$	$E_6^{\oplus 2} \subset E_7^{\oplus 2}$	$A_2 \subset D_{10}$		$(-6)^{\oplus 2} \oplus A_2^{\perp D_{10}}$	D_7	$\mathbb{Z}^3$
2.3	$E_6^{\oplus 4}$	$E_6^{\oplus 2} \subset E_6^{\oplus 2}$	$A_2 \subset E_6$		$E_6 \oplus A_2^{\oplus 2}$	$E_6 \oplus A_2^{\oplus 2}$	0

3. $\mathrm{NS}(X) = U \oplus E_8 \oplus A_2$, $T_X = U^{\oplus 2} \oplus E_6$, $T_0 = E_8 \oplus E_6$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
3.1	$E_8^{\oplus 3}$	$E_8 \subset E_8$	$E_6 \subset E_8$		$E_8 \oplus A_2$	$E_8 \oplus A_2$	0

4. $\mathrm{NS}(X) = U \oplus E_6 \oplus A_2^{\oplus 3}$, $T_X = U \oplus U(3) \oplus A_2^{\oplus 2}$, $T_0 = E_6 \oplus A_2^{\oplus 3}$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
4.1	$E_8^{\oplus 3}$	$E_6 \subset E_8$	$A_2^{\oplus 2} \subset E_8$	$A_2 \subset E_8$	$E_6 \oplus A_2^{\oplus 3}$	$E_6 \oplus A_2^{\oplus 3}$	0
4.2	$E_8 \oplus D_{16}$	$E_6 \subset E_8$	$A_2^{\oplus} \subset D_{16}$		$A_2 \oplus (A_2^{\oplus 3})^{\perp D_{16}}$	$D_7 \oplus A_2$	$\mathbb{Z}^3$
4.3	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \oplus E_7$	$A_2 \oplus D_{10}$		$(-6) \oplus E_7 \oplus (A_2^{\oplus 3})^{\perp D_{10}}$	E_7	$\mathbb{Z}^5$

4.4	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2 \subset E_7$	$A_2^{\oplus 2} \subset D_{10}$	$(-6) \oplus A_5 \oplus (A_2^{\oplus 2})^{\perp D_{10}}$	$D_4 \oplus A_5$	$\mathbb{Z}^3$
4.5	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2^{\oplus 2} \subset E_7$	$A_2 \subset D_{10}$	$(-6)^{\oplus} \oplus A_2 \oplus A_2^{\perp D_{10}}$	$D_7 \oplus A_2$	$\mathbb{Z}^3$
4.6	$E_7 \oplus A_{17}$	$E_6 \subset E_7$	$A_2^{\oplus 3} \subset A_{17}$		$(-6) \oplus (A_2^{\oplus 3})^{\perp A_{17}}$	A_8	$\mathbb{Z}^4$
4.7	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset E_6$	$A_2 \subset E_6$	$E_6 \oplus A_2^{\oplus 3}$	$E_6 \oplus A_2^{\oplus 3}$	0
4.8	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$(A_2)^{\oplus 3} \subset E_6^{\oplus 3}$		$A_2^{\oplus 6}$	$A_2^{\oplus 6}$	$\mathbb{Z}/3$
4.9	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 3} \subset A_{11}$		$D_7 \oplus (A_2^{\oplus})^{\perp A_{11}}$	$D_7 \oplus A_2$	$\mathbb{Z}^3$
4.10	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2 \subset D_7$	$A_2^{\oplus 2} \subset A_{11}$	$A_2^{\perp D_7} \oplus (A_2^{\oplus 2})^{\perp A_{11}}$	$D_4 \oplus A_5$	$\mathbb{Z}^3$
4.11	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset D_7$	$A_2 \subset A_{11}$	$(A_2^{\oplus 2})^{\perp D_7} \oplus A_2^{\perp A_{11}}$	A_8	$\mathbb{Z}^4$

5. $\mathrm{NS}(X) = U \oplus E_6^{\oplus 2}$, $T_X = U^{\oplus 2} \oplus A_2^{\oplus 2}$, $T_0 = E_8 \oplus A_2^{\oplus 2}$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
5.1	$E_8^{\oplus 3}$	$E_8 \subset E_8$	$A_2^{\oplus 2} \subset E_8$		$E_8 \oplus A_2^{\oplus 2}$	$E_8 \oplus A_2^{\oplus 2}$	0
5.2	$E_8^{\oplus 3}$	$E_8 \subset E_8$	$(A_2)^{\oplus 2} \subset E_8^{\oplus 2}$		$E_6^{\oplus 2}$	$E_6^{\oplus 2}$	0
5.3	$E_8 \oplus D_{16}$	$E_8 \subset E_8$	$A_2^{\oplus 2} \subset D_{16}$		$(A_2^{\oplus 2})^{\perp D_{16}}$	D_{10}	$\mathbb{Z}^2$

6. $\mathrm{NS}(X) = U \oplus E_6^{\oplus 2} \oplus A_2$, $T_X = U \oplus U(3) \oplus A_2$, $T_0 = E_6 \oplus A_2^{\oplus 2}$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
6.1	$E_8^{\oplus 3}$	$E_6 \subset E_8$	$A_2^{\oplus 2} \subset E_8$		$E_8 \oplus A_2^{\oplus 3}$	$E_8 \oplus A_2^{\oplus 3}$	0
6.2	$E_8^{\oplus 3}$	$E_6 \subset E_8$	$(A_2)^{\oplus 2} \subset E_8^{\oplus 2}$		$E_6^{\oplus 2} \oplus A_2$	$E_6^{\oplus 2} \oplus A_2$	0
6.3	$E_8 \oplus D_{16}$	$E_6 \subset E_8$	$A_2^{\oplus 2} \subset D_{16}$		$A_2 \oplus (A_2^{\oplus 2})^{\perp D_{16}}$	$D_{10} \oplus A_2$	$\mathbb{Z}^2$
6.4	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2^{\oplus 2} \subset E_7$		$(-6)^{\oplus 2} \oplus D_{10} \oplus A_2$	$D_{10} \oplus A_2$	$\mathbb{Z}^2$
6.5	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2 \subset E_7$	$A_2 \subset D_{10}$	$(-6) \oplus A_5 \oplus A_2^{\perp D_{10}}$	$D_7 \oplus A_5$	$\mathbb{Z}^2$
6.6	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2^{\oplus 2} \subset D_{10}$		$(-6) \oplus E_7 \oplus (A_2^{\oplus 2})^{\perp D_{10}}$	$E_7 \oplus D_4$	$\mathbb{Z}^3$
6.7	$E_7 \oplus A_{17}$	$E_6 \subset E_7$	$A_2^{\oplus 2} \subset A_{17}$		$(-6) \oplus (A_2^{\oplus 2})^{\perp A_{17}}$	A_{11}	$\mathbb{Z}^3$
6.8	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$A_2^{\oplus} \subset E_6$		$E_6^{\oplus 2} \oplus A_2$	$E_6^{\oplus 2} \oplus A_2$	0
6.9	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$A_2 \subset E_6$	$A_2 \subset E_6$	$E_6 \oplus A_2^{\oplus 4}$	$E_6 \oplus A_2^{\oplus 4}$	$\mathbb{Z}/3$
6.10	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset D_7$		$(A_2^{\oplus 2})^{\perp D_7} \oplus A_{11}$	A_{11}	$\mathbb{Z}^3$
6.11	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2 \subset D_7$	$A_2 \subset A_{11}$	$A_2^{\perp D_7} \oplus A_2^{\perp A_{11}}$	$D_4 \oplus A_8$	$\mathbb{Z}^2$
6.12	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2^{\oplus 2} \subset A_{11}$		$D_7 \oplus (A_2^{\oplus 2})^{\perp A_{11}}$	$D_7 \oplus A_5$	$\mathbb{Z}^2$

7. $\mathrm{NS}(X) = U \oplus E_6 \oplus E_8$, $T_X = U^{\oplus 2} \oplus A_2$, $T_0 = E_8 \oplus A_2$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
7.1	$E_8^{\oplus 3}$	$E_8 \subset E_8$	$A_2 \subset E_8$		$E_6 \oplus E_8$	$E_6 \oplus E_8$	0
7.2	$E_8 \oplus D_{16}$	$E_8 \subset E_8$	$A_2 \subset D_{16}$		$A_2^{\perp D_{16}}$	D_{13}	$\mathbb{Z}$

8. $\mathrm{NS}(X) = U \oplus E_6 \oplus E_8 \oplus A_2$, $T_X = U \oplus U(3)$, $T_0 = E_6 \oplus A_2$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
8.1	$E_8^{\oplus 3}$	$E_6 \subset E_8$	$A_2 \subset E_8$		$E_6 \oplus E_8 \oplus A_2$	$E_6 \oplus E_8 \oplus A_2$	0
8.2	$E_8 \oplus D_{16}$	$E_6 \subset E_8$	$A_2 \subset D_{16}$		$A_2 \oplus A_2^{\perp D_{16}}$	$D_{13} \oplus A_2$	$\mathbb{Z}$
8.3	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2 \subset E_7$		$(-6) \oplus A_5 \oplus D_{10}$	$D_{10} \oplus A_5$	$\mathbb{Z} \oplus \mathbb{Z}/2$
8.4	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$	$A_2 \subset D_{10}$		$(-6) \oplus E_7 \oplus A_2^{\perp D_{10}}$	$E_7 \oplus D_7$	$\mathbb{Z}^2$
8.5	$E_7 \oplus A_{17}$	$E_6 \subset E_7$	$A_2 \subset A_{17}$		$(-6) \oplus A_2^{\perp A_{17}}$	A_{14}	$\mathbb{Z}^2$
8.6	$E_6^{\oplus 4}$	$E_6 \subset E_6$	$A_2 \subset E_6$		$E_6^{\oplus 2} \oplus A_2^{\oplus 2}$	$E_6^{\oplus 2} \oplus A_2^{\oplus 2}$	$\mathbb{Z}/3$
8.7	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2 \subset D_7$		$A_2^{\perp D_7} \oplus A_{11}$	$D_4 \oplus A_{11}$	$\mathbb{Z}$
8.8	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$	$A_2 \subset A_{11}$		$D_7 \oplus A_2^{\perp A_{11}}$	$D_7 \oplus A_8$	$\mathbb{Z}$

9. $\mathrm{NS}(X) = U \oplus E_8^{\oplus 2}$, $T_X = U^{\oplus 2}$, $T_0 = E_8$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
9.1	$E_8^{\oplus 3}$	$E_8 \subset E_8$			$E_8^{\oplus 2}$	$E_8^{\oplus 2}$	0
9.2	$E_8 \oplus D_{16}$	$E_8 \subset E_8$			D_{16}	D_{16}	$\mathbb{Z}/2$

10. $\mathrm{NS}(X) = U \oplus E_8^{\oplus 2} \oplus A_2$, $T_X = A_2(-1)$, $T_0 = E_6$

No.	L_{root}	Embeddings			M	$T = M_{\mathrm{root}}$	$\mathrm{MW}(\pi)$
10.1	$E_8^{\oplus 3}$	$E_6 \subset E_8$			$E_8^{\oplus 2} \oplus A_2$	$E_8^{\oplus 2} \oplus A_2$	0
10.2	$E_8 \oplus D_{16}$	$E_6 \subset E_8$			$D_{16} \oplus A_2$	$D_{16} \oplus A_2$	$\mathbb{Z}/2$
10.3	$E_7^{\oplus 2} \oplus D_{10}$	$E_6 \subset E_7$			$(-6) \oplus E_7 \oplus D_{10}$	$E_7 \oplus D_{10}$	$\mathbb{Z} \oplus \mathbb{Z}/2$
10.4	$E_7 \oplus A_{17}$	$E_6 \subset E_7$			$(-6) \oplus A_{17}$	A_{17}	$\mathbb{Z} \oplus \mathbb{Z}/3$
10.5	$E_6^{\oplus 4}$	$E_6 \subset E_6$			$E_6^{\oplus 3}$	$E_6^{\oplus 3}$	$\mathbb{Z}/3$
10.6	$E_6 \oplus D_7 \oplus A_{11}$	$E_6 \subset E_6$			$D_7 \oplus A_{11}$	$D_7 \oplus A_{11}$	$\mathbb{Z}/4$

3.4.1 Determining T_0

The first step of the Kneser–Nishiyama method consists of finding a lattice T_0 of root type such that $\operatorname{rank}(T_0) = \operatorname{rank}(T_X) + 4$, $G_{T_0} = G_{T_X}$ and $q_{T_0} = q_{T_X}$. Table 7 shows the choice explicitly for each T_X in Table 5.

No.	T_X	G_{T_X}	T_0
1	$U \oplus U(3) \oplus A_2^{\oplus 3}$	$(\mathbb{Z}/3\mathbb{Z})^5$	$E_6 \oplus A_2^{\oplus 4}$
2	$U \oplus U(3) \oplus E_6$	$(\mathbb{Z}/3\mathbb{Z})^3$	$E_6^{\oplus 2} \oplus A_2$
3	$U^{\oplus 2} \oplus E_6$	$\mathbb{Z}/3\mathbb{Z}$	$E_8 \oplus E_6$
4	$U \oplus U(3) \oplus A_2^{\oplus 2}$	$(\mathbb{Z}/3\mathbb{Z})^4$	$E_6 \oplus A_2^{\oplus 3}$
5	$U^{\oplus 2} \oplus A_2^{\oplus 2}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$E_8 \oplus A_2^{\oplus 2}$
6	$U \oplus U(3) \oplus A_2$	$(\mathbb{Z}/3\mathbb{Z})^3$	$E_6 \oplus A_2^{\oplus 2}$
7	$U^{\oplus 2} \oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$	$E_8 \oplus A_2$
8	$U \oplus U(3)$	$(\mathbb{Z}/3\mathbb{Z})^2$	$E_6 \oplus A_2$
9	$U^{\oplus 2}$	$\{e\}$	E_8
10	$A_2(-1)$	$\mathbb{Z}/3\mathbb{Z}$	E_6

Tabela 7 – T_0 for each surface X

Proposition 3.4.3. *For every T_X, T_0 in Table 7, $G_{T_X} = G_{T_0}$ and $q_{T_0} = q_{T_X}$.*

Demonstração. Firstly, observe that for any L_1, L_2, we have $G_{L_1 \oplus L_2} = G_{L_1} \times G_{L_2}$ and $q_{L_1 \oplus L_2} = q_{L_1} + q_{L_2}$. This reduces the required calculations to the cases $T_X = A_2(-1), U^{\oplus 2}, U \oplus U(3)$ and $T_0 = E_6, E_8, E_6 \oplus A_2$, respectively. The lattices G_{T_X} are given in (ARTEBANI; SARTI, 2008, Lemma 1.3, Table 1), and both G_{T_0} and q_{T_0} can be calculated using (NISHIYAMA, 1996, Lemma 1.2).

Firstly, let $T_X = U^{\oplus 2}$. Since T_X is unimodular, its discriminant group G_{T_X} is trivial, and consequently $q_{T_X} = 0$.

Let $T_X = A_2(-1)$, and write its generators as a_1, a_2, with $a_1^2 = a_2^2 = 2$, $a_1 \cdot a_2 = -1$. Then the discriminant group G_{T_X} is generated by

$$w = \frac{2}{3}a_1 + \frac{1}{3}a_2.$$

Furthermore, we calculate its discriminant lattice

$$q_{T_X}(w) \equiv \frac{2}{3} \equiv -\frac{4}{3} \mod 2\mathbb{Z}.$$

Finally, let $T_X = U \oplus U(3)$, and write its generators as u_1, u_2, u_1', u_2', with $u_i^2 = (u_i')^2 = u_i \cdot u_j' = 0$, $u_1 \cdot u_2 = 1$, and $u_1' \cdot u_2' = 3$. Then G_{T_X} is generated by

$$w_1 = \frac{1}{3}u_1' + \frac{1}{3}u_2'$$
$$w_2 = \frac{2}{3}u_1' + \frac{1}{3}u_2'.$$

Then, we calculate

$$q_{T_X}(w_1) \equiv -\frac{4}{3} \mod 2\mathbb{Z}$$
$$q_{T_X}(w_2) \equiv -\frac{2}{3} \mod 2\mathbb{Z}.$$

Using (NISHIYAMA, 1996, Lemma 1.2), we can check that the values of G_{T_0} and q_{T_0} agree with the ones calculated above. □

3.4.2 Embeddings into Niemeier Lattices

In this section, we show how to calculate the primitive embeddings of each T_0 in Table 7 into a Niemeier lattices L. Firstly, we notice that since every T_0 has a sublattice of type E_6 or E_8, every Niemeier lattice L which allows an embedding $\varphi\colon T_0 \to L$ must also have a sublattice of type E_n. These lattices are shown in Table 8.

L_{root}	L/L_{root}
$E_8^{\oplus 3}$	0
$E_8 \oplus D_{16}$	$\mathbb{Z}/2\mathbb{Z}$
$E_7^{\oplus 2} \oplus D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$E_7 \oplus A_{17}$	$\mathbb{Z}/6\mathbb{Z}$
$E_6^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^2$
$E_6 \oplus D_7 \oplus A_{11}$	$\mathbb{Z}/12\mathbb{Z}$

Tabela 8 – Niemeier lattices containing an E_n sublattice

For any lattice L in Table 8, we can calculate the primitive embeddings of $A_2^{\oplus \ell}$, E_6 and E_8 into the components of L_{root}. We obtain an embedding $\varphi\colon T_0 \to L_{\mathrm{root}}$ by composing these embeddings together. Similarly, we calculate the orthogonal $\varphi(T_0)^{\perp L_{\mathrm{root}}}$ by composing the orthogonal lattices for each component of L_{root}.

The primitive embeddings of A_2, E_6 and E_8 into another root lattice are shown in (NISHIYAMA, 1996, Lemmas 4.1, 4.2 and 4.3), and their corresponding orthogonal lattices in (NISHIYAMA, 1996, Corollary 4.4). Furthermore, every possible embedding of a root lattice into a lattice of type E_n is shown in (NISHIYAMA, 1996, Table 5.1).

We move to the calculation of primitive embeddings of $A_2^{\oplus \ell}$ into A_n or D_n, for $\ell \geq 2$, and their orthogonal complements. Firstly, we prove the following lemma.

Lemma 3.4.4. *If N_1, N_2 are sublattices of a lattice L, N_2 is of root type and $N_2 \subseteq N_1^{\perp L}$, then $((N_1 \oplus N_2)^{\perp L})_{\text{root}} = (N_2^{\perp (N_1^{\perp L})_{\text{root}}})_{\text{root}}$.*

Demonstração. Since N_2 is of root type, $N_2 \subseteq (N_1^{\perp L})_{\text{root}} \subseteq N_1^{\perp L}$. Then,

$$N_2^{\perp (N_1^{\perp L})_{\text{root}}} \subseteq N_2^{\perp (N_1^{\perp L})} = (N_1 \oplus N_2)^{\perp L}.$$

Taking the root type of both sides, we get $(N_2^{\perp (N_1^{\perp L})_{\text{root}}})_{\text{root}} \subseteq ((N_1 \oplus N_2)^{\perp L})_{\text{root}}$. Now suppose $x \in ((N_1 \oplus N_2)^{\perp L})_{\text{root}}$. Then, x is generated by roots and $\langle x, n_1 \rangle = 0$ for all $n_1 \in N_1$, so by definition $x \in (N_1^{\perp L})_{\text{root}}$. Since $\langle x, n_2 \rangle = 0$ for all $n_2 \in N_2$, it is true that $x \in (N_2^{\perp (N_1^{\perp L})_{\text{root}}})_{\text{root}}$. This gets us to the result. □

Proposition 3.4.5. *The primitive embeddings of $A_2^{\oplus \ell}$ in A_n or D_n, up to an action of their Weyl group, are as follows.*

i) There is a unique embedding given by $A_2^{\oplus \ell} = \bigoplus_{i=0}^{\ell-1} \langle a_{3i+1}, a_{3i+2} \rangle \subset A_n$ for $n \geq 3\ell - 1$. Furthermore, the orthogonal of this embedding is

$$(A_2^{\oplus \ell})_{\text{root}}^{\perp A_n} = \begin{cases} 0 & \text{if } 3\ell - 1 \leq n \leq 3\ell, \\ A_{n-3\ell} & \text{if } n \geq 3\ell + 1. \end{cases}$$

ii) There is a unique embedding given by $A_2^{\oplus \ell} = \bigoplus_{i=1}^{\ell} \langle d_{3\ell-1}, d_{3\ell} \rangle \subset D_n$ for $n \geq 3\ell$. Furthermore, the orthogonal of this embedding is

$$(A_2^{\oplus \ell})_{\text{root}}^{\perp D_n} = \begin{cases} 0 & \text{if } 3\ell \leq n \leq 3\ell + 1, \\ A_1^{\oplus 2} & \text{if } n = 3\ell + 2, \\ A_3 & \text{if } n = 3\ell + 3, \\ D_{n-3\ell} & \text{if } n \geq 3\ell + 4. \end{cases}$$

Demonstração. We prove this by induction. When $\ell = 1$, the primitive embeddings of A_2 into A_n and D_n are proved in (NISHIYAMA, 1996, Lemma 4.1, Lemma 4.2), and their respective orthogonal lattices are calculated in (NISHIYAMA, 1996, Corollary 4.4). Suppose this is true for ℓ. Then, for $n \geq 3\ell + 2$, there is a unique primitive embedding of $A_2^{\oplus \ell}$ in A_n, and the orthogonal lattice is $A_{n-3\ell} = \langle a_{3\ell+1}, ..., a_n \rangle$. We know that $A_2 = \langle a_{3\ell+1}, a_{3\ell+2} \rangle$ is the unique primitive embedding of A_2 in $A_{n-3\ell}$, up to an action of $W(A_{n-3\ell}) \subset W(A_n)$. Gluing together both embeddings, we obtain a unique primitive embedding of $A_2^{\oplus \ell+1}$ in A_n up to an action of $W(A_n)$. By Lemma 3.4.4, $(A_2^{\oplus \ell})_{\text{root}}^{\perp A_n} = (A_2^{\perp A_{n-3\ell}})_{\text{root}}$. This is equal to 0 if $n = 3\ell + 2$ or $3\ell + 3$, and to $A_{n-3\ell-3}$ if $n \geq 3\ell + 4$.

For $n \geq 3\ell + 3$, we know that there is a unique primitive embedding of $A_2^{\oplus \ell}$ in D_n, and the orthogonal is equal to A_3 if $n = 3\ell + 3$, and to $D_{n-3\ell}$ if $n \geq 3\ell + 4$. In either case, there is a unique primitive embedding of A_2 up to an action of the Weyl group, thus obtaining an embedding of $A_2^{\oplus \ell+1}$ in D_n. We can apply Lemma 3.4.4 to obtain the result. □

Using this result, we are able to explicitly show the generators of orthogonal complements of primitive embeddings of a lattice N into a Niemeier lattice L, as well as their root types, when N is a sum of A_2, E_6 and E_8.

Firstly, we establish some notation. Let a_i, d_i and e_i denote the canonical generators of the A_n, D_n and E_n lattices, respectively. For ease of notation, we define elements α_i in lattices of type A_n as follows.

$$\alpha_i = a_{3i-2} + 2a_{3i-1} + 3a_{3i} + 2a_{3i+1} + a_{3i+2},$$
$$\alpha_i' = a_{3i-2} + 2a_{3i-1} + 3a_{3i}.$$

Then, letting $\delta_0 = d_1$, we define the following elements of D_n recursively.

$$\delta_i = \delta_{i-1} + d_{3i-1} + 2d_{3i} + 2d_{3i+1} + d_{3i+2},$$
$$\delta_i' = 2\delta_{i-1} + d_{3i-1} + d_{3_i} + d_{3i+1},$$
$$\delta_i'' = \delta_{i-1} + d_{3i-1} + 2d_{3i} + 2d_{3i+1}.$$

Finally, we denote a general element of E_n as follows.

$$\begin{aligned}
\overset{\lambda_1}{\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6} &= \sum_{i=1}^{6} \lambda_i e_i,\\
\overset{\lambda_1}{\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7} &= \sum_{i=1}^{7} \lambda_i e_i,\\
\overset{\lambda_1}{\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\lambda_7\lambda_8} &= \sum_{i=1}^{8} \lambda_i e_i.
\end{aligned}$$

In Table 9, for pairs of lattices N, L, we explicitly present the unique primitive embedding $\varphi\colon N \to L$ (up to an action of the Weyl group of L), the orthogonal complement $\varphi(N)^{\perp L}$ and its root type $(\varphi(N)^{\perp L})_{\text{root}}$.

Tabela 9 – Primitive embeddings and orthogonal complements

L	N	$\varphi\colon N \to L$	$\varphi(N)^{\perp L}$	$(\varphi(N)^{\perp L})_{\text{root}}$
A_{11}	A_2	$\langle a_1, a_2\rangle$	$\begin{pmatrix} -12 & 3 & 0 & \cdots & 0 \\ 3 & & & & \\ 0 & & A_8 & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} = \langle \alpha_1', a_4, ..., a_{11}\rangle$	A_8
	$A_2^{\oplus 2}$	$\langle a_1, a_2\rangle \oplus \langle a_4, a_5\rangle$	$\begin{pmatrix} -6 & 3 & 0 & 0 & \cdots & 0 \\ 3 & -12 & 3 & 0 & \cdots & 0 \\ 0 & 3 & & & & \\ 0 & 0 & & A_5 & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix} = \langle \alpha_1, \alpha_2', a_7, ..., a_{11}\rangle$	A_5
	$A_2^{\oplus 3}$	$\langle a_1, a_2\rangle \oplus \langle a_4, a_5\rangle \oplus \langle a_7, a_8\rangle$	$\begin{pmatrix} -6 & 3 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ 0 & 3 & -12 & 3 & 0 \\ 0 & 0 & 3 & & \\ 0 & 0 & 0 & \multicolumn{2}{c}{A_2} \end{pmatrix} = \langle \alpha_1, \alpha_2, \alpha_3', a_{10}, a_{11}\rangle$	A_2
	$A_2^{\oplus 4}$	$\bigoplus_{i=0}^{3} \langle a_{3i+1}, a_{3i+2}\rangle$	$A_3(3) = \begin{pmatrix} -6 & 3 & 0 \\ 3 & -6 & 3 \\ 0 & 3 & -6 \end{pmatrix} = \langle \alpha_1, \alpha_2, \alpha_3\rangle$	0
A_{17}	A_2	$\langle a_1, a_2\rangle$	$\begin{pmatrix} -12 & 3 & 0 & \cdots & 0 \\ 3 & & & & \\ 0 & & A_{14} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} = \langle \alpha_1', a_4, ..., a_{17}\rangle$	A_{14}

<table>
<tr><th>L</th><th>N</th><th>$\varphi\colon N \to L$</th><th>$\varphi(N)^{\perp L}$</th><th>$(\varphi(N)^{\perp L})_{\mathrm{root}}$</th></tr>
<tr><td>A_{17}</td><td>$A_2^{\oplus 2}$</td><td>$\langle a_1, a_2\rangle \oplus \langle a_4, a_5\rangle$</td><td>$\left(\begin{array}{cc|cccc} -6 & 3 & 0 & 0 & \cdots & 0 \\ 3 & -12 & 3 & 0 & \cdots & 0 \\ \hline 0 & 3 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & A_{11} & & \\ 0 & 0 & & & & \end{array}\right) = \langle \alpha_1, \alpha_2', a_7, ..., a_{17}\rangle$</td><td>$A_{11}$</td></tr>
<tr><td></td><td>$A_2^{\oplus 3}$</td><td>$\langle a_1, a_2\rangle \oplus \langle a_4, a_5\rangle \oplus \langle a_7, a_8\rangle$</td><td>$\left(\begin{array}{ccc|cccc} -6 & 3 & 0 & 0 & 0 & \cdots & 0 \\ 3 & -6 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 3 & -12 & 3 & 0 & \cdots & 0 \\ \hline 0 & 0 & 3 & & & & \\ 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & & A_8 & & \\ 0 & 0 & 0 & & & & \end{array}\right) = \langle \alpha_1, \alpha_2, \alpha_3', a_{10}, ..., a_{17}\rangle$</td><td>$A_8$</td></tr>
<tr><td></td><td>$A_2^{\oplus 4}$</td><td>$\bigoplus_{i=0}^{3} \langle a_{3i+1}, a_{3i+2}\rangle$</td><td>$\left(\begin{array}{cccc|cccc} -6 & 3 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 3 & -6 & 3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 3 & -6 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & -12 & 3 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 3 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & & A_5 & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right) = \begin{array}{c}\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4', \\ a_{13}, ..., a_{17}\rangle\end{array}$</td><td>$A_5$</td></tr>
<tr><td></td><td>$A_2^{\oplus 5}$</td><td>$\bigoplus_{i=0}^{4} \langle a_{3i+1}, a_{3i+2}\rangle$</td><td>$\left(\begin{array}{ccccc|cc} -6 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -12 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & \multicolumn{2}{c}{\multirow{2}{*}{A_2}} \\ 0 & 0 & 0 & 0 & 0 & & \end{array}\right) = \begin{array}{c}\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \\ \alpha_5', a_{16}, a_{17}\rangle\end{array}$</td><td>$A_2$</td></tr>
<tr><td></td><td>$A_2^{\oplus 6}$</td><td>$\bigoplus_{i=0}^{5} \langle a_{3i+1}, a_{3i+2}\rangle$</td><td>$A_5(3) = \begin{pmatrix} -6 & 3 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 3 & -6 \end{pmatrix} = \begin{array}{c}\langle \alpha_1, \alpha_2, \alpha_3, \\ \alpha_4, \alpha_5\rangle\end{array}$</td><td>$0$</td></tr>
<tr><td>D_7</td><td>A_2</td><td>$\langle d_2, d_3\rangle$</td><td>$\left(\begin{array}{c|cccc} -4 & -1 & 1 & 0 & 0 \\ \hline -1 & & & & \\ 1 & & D_4 & & \\ 0 & & & & \\ 0 & & & & \end{array}\right) = \langle \delta_1', \delta_1, d_5, d_6, d_7\rangle$</td><td>$D_4$</td></tr>
<tr><td></td><td>$A_2^{\oplus 2}$</td><td>$\langle d_2, d_3\rangle \oplus \langle d_5, d_6\rangle$</td><td>$\begin{pmatrix} -4 & -1 & 0 \\ -1 & -4 & -2 \\ 0 & -2 & -4 \end{pmatrix} = \langle \delta_1', \delta_2', \delta_2''\rangle$</td><td>$0$</td></tr>
<tr><td>D_{10}</td><td>A_2</td><td>$\langle d_2, d_3\rangle$</td><td>$\left(\begin{array}{c|ccccc} -4 & -1 & 1 & 0 & \cdots & 0 \\ \hline -1 & & & & & \\ 1 & & & & & \\ 0 & & & D_7 & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right) = \langle \delta_1', \delta_1, d_5, \ldots, d_{10}\rangle$</td><td>$D_7$</td></tr>
<tr><td></td><td>$A_2^{\oplus 2}$</td><td>$\langle d_2, d_3\rangle \oplus \langle d_5, d_6\rangle$</td><td>$\left(\begin{array}{cc|cccc} -4 & -1 & 0 & 0 & 0 & 0 \\ -1 & -4 & -1 & 1 & 0 & 0 \\ \hline 0 & -1 & & & & \\ 0 & 1 & & D_4 & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{array}\right) = \begin{array}{c}\langle \delta_1', \delta_2', \delta_2, \\ d_8, d_9, d_{10}\rangle\end{array}$</td><td>$D_4$</td></tr>
<tr><td></td><td>$A_2^{\oplus 3}$</td><td>$\langle d_2, d_3\rangle \oplus \langle d_5, d_6\rangle \oplus \langle d_8, d_9\rangle$</td><td>$\begin{pmatrix} -4 & -1 & 0 & 0 \\ -1 & -4 & -1 & 0 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & -2 & -4 \end{pmatrix} = \langle \delta_1', \delta_2', \delta_3', \delta_3''\rangle$</td><td>$0$</td></tr>
</table>

L	N	$\varphi\colon N \to L$	$\varphi(N)^{\perp L}$	$(\varphi(N)^{\perp L})_{\mathrm{root}}$
D_{16}	A_2	$\langle d_2, d_3\rangle$	$\left(\begin{array}{c\|c} -4 & -1\ \ 1\ \ 0\ \cdots\ 0 \\ \hline -1 & \\ 1 & \\ 0 & D_{13} \\ \vdots & \\ 0 & \end{array}\right) = \langle \delta_1', \delta_1, d_5, \ldots, d_{16}\rangle$	D_{13}
	$A_2^{\oplus 2}$	$\langle d_2, d_3\rangle \oplus \langle d_5, d_6\rangle$	$\left(\begin{array}{cc\|ccccc} -4 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -4 & -1 & 1 & 0 & \cdots & 0 \\ \hline 0 & -1 & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & & & D_{10} & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & & & & & \end{array}\right) = \langle \delta_1', \delta_2', \delta_2, d_8, \ldots, d_{16}\rangle$	D_{10}
	$A_2^{\oplus 3}$	$\langle d_2, d_3\rangle \oplus \langle d_5, d_6\rangle \oplus \langle d_8, d_9\rangle$	$\left(\begin{array}{ccc\|ccccc} -4 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & -4 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -4 & -1 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & -1 & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & D_7 & & \\ \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & & & & & \end{array}\right) = \langle \delta_1', \delta_2', \delta_3', \delta_3, d_{11}, \ldots, d_{16}\rangle$	D_7
	$A_2^{\oplus 4}$	$\bigoplus_{i=1}^4 \langle d_{3i-1}, d_{3i}\rangle$	$\left(\begin{array}{cccc\|cccc} -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -4 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & & & & \\ 0 & 0 & 0 & 1 & & D_4 & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \end{array}\right) = \langle \delta_1', \delta_2', \delta_3', \delta_4' \ \delta_4, d_{14}, d_{15}, d_{16}\rangle$	D_4
	$A_2^{\oplus 5}$	$\bigoplus_{i=1}^5 \langle d_{3i-1}, d_{3i}\rangle$	$\begin{pmatrix} -4 & -1 & 0 & 0 & 0 & 0 \\ -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & -1 & -4 & -1 & 0 & 0 \\ 0 & 0 & -1 & -4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -4 & -2 \\ 0 & 0 & 0 & 0 & -2 & -4 \end{pmatrix} = \langle \delta_1', \delta_2', \delta_3', \delta_4', \delta_5'', \delta_5''\rangle$	0
E_6	A_2	$\langle e_2, e_3\rangle$	$A_2^{\oplus 2} = \langle e_5, e_6\rangle \oplus \langle e_1, \substack{2\\12321}\rangle$	$A_2^{\oplus 2}$
	$A_2^{\oplus 2}$	$\langle e_2, e_3\rangle \oplus \langle e_5, e_6\rangle$	$A_2 = \langle e_1, \substack{2\\12321}\rangle$	A_2
	E_6	$\langle e_1, ..., e_6\rangle$	0	0
E_7	A_2	$\langle e_2, e_3\rangle$	$A_5 = \langle e_5, e_6, e_7, \substack{2\\123210}, e_1\rangle$	A_5
	$A_2^{\oplus 2}$	$\langle e_2, e_3\rangle \oplus \langle e_5, e_6\rangle$	$(-6) \oplus A_2 = \langle \substack{3\\246543}\rangle \oplus \langle e_1, \substack{2\\123210}\rangle$	A_2
	E_6	$\langle e_1, ..., e_6\rangle$	$(-6) = \langle \substack{3\\246543}\rangle$	0
E_8	A_2	$\langle e_2, e_3\rangle$	$E_6 = \langle e_8, e_5, e_6, e_7, \substack{2\\1232100}, e_1\rangle$	E_6
	$A_2^{\oplus 2}$	$\langle e_2, e_3\rangle \oplus \langle e_5, e_6\rangle$	$A_2^{\oplus 2} = \langle e_1, \substack{2\\1232100}\rangle \oplus \langle e_8, \substack{3\\2465432}\rangle$	$A_2^{\oplus 2}$
	E_6	$\langle e_1, ..., e_6\rangle$	$A_2 = \langle e_8, \substack{3\\2465432}\rangle$	A_2

3.4.3 Calculation of the torsion subgroups

In this section, we calculate the torsion subgroup of the Mordell–Weil group of each fibration in Table 6. In (SHIMADA, 2000, Table 1), the possible torsion subgroups are presented for each ADE-type. For most of the fibrations in Table 6, there is only one admissible torsion subgroup, and thus there is no need to calculate $\overline{W_{\text{root}}}/W_{\text{root}}$ explicitly. The fibrations with multiple admissible torsion subgroups are 4.8, 6.9, 8.3, 8.6, 8.7, 9.2, 10.3 and 10.4. When $\operatorname{rank}\mathrm{MW}(\pi) = 0$, the torsion subgroup is isomorphic to W/M (see (NISHIYAMA, 1996, Lemma 6.6)), and the torsion can be determined by calculating $\det T_0 = \det W$ and $\det M$. This allows us to determine the torsion for fibrations 4.8, 6.9, 8.6 and 9.2. For 10.3 and 10.4, the torsion subgroups were already calculated in (NISHIYAMA, 1996, Theorem 3.1, Table 1.1).

Proposition 3.4.6. *The torsion subgroups of fibrations 8.3 and 8.7 in Table 6 are* $\mathbb{Z}/2\mathbb{Z}$ *and* 0*, respectively.*

Demonstração. We start with fibration 8.3. By ((SHIMADA, 2000, Table 1)), the torsion subgroup is either $\mathbb{Z}/2\mathbb{Z}$ or 0. We determine that the subgroup must be $\mathbb{Z}/2\mathbb{Z}$ by explicitly showing an order 2 element in $\overline{W_{\text{root}}}/W_{\text{root}}$. The lattice T_0 is embedded in the Niemeier lattice $L_{\text{root}} = E_7^{\oplus 2} \oplus D_{10}$. Let $e_1^{(1)}, \ldots, e_7^{(1)}$, $e_1^{(2)}, \ldots, e_7^{(2)}$, $d_1, \ldots, d_{10}$ be generators for L_{root}. Then, T_0 is embedded isomorphically onto the sublattice $\langle e_1^{(1)}, ..., e_6^{(1)} \rangle \oplus \langle e_2^{(2)}, e_3^{(2)} \rangle \subset L_{\text{root}}$. We calculate W_{root} by taking the root type of the orthogonal complement of $\varphi(T_0)$, obtaining

$$W_{\text{root}} = \langle e_5^{(2)}, e_6^{(2)}, e_7^{(2)}, (2e_1^{(2)}{+}e_2^{(2)}{+}2e_3^{(2)}{+}3e_4^{(2)}{+}2e_5^{(2)}{+}e_6^{(2)}), e_1^{(2)} \rangle \oplus \langle d_1, ..., d_{10} \rangle.$$

Then, we can find an element $\eta \in L \setminus L_{\text{root}}$ such that $2\eta \in W_{\text{root}}$. Therefore, η lies in the primitive closure $\overline{W_{\text{root}}}$, and the torsion subgroup is $\mathbb{Z}/2\mathbb{Z}$. Explicitly,

$$\eta = \frac{e_1^{(2)} + e_5^{(2)} + e_7^{(2)} + d_1 + d_4 + d_6 + d_8 + d_{10}}{2}.$$

Now consider fibration 8.7. By ((SHIMADA, 2000, Table 1)), the torsion subgroup is also given by either $\mathbb{Z}/2\mathbb{Z}$ or 0. We determine that the subgroup must be trivial by showing that no element of order 2 in L/L_{root} is in $\overline{W_{\text{root}}}$. Let $L_{\text{root}} = E_6 \oplus D_7 \oplus A_{11}$ and $e_1, \ldots, e_6$, $d_1, \ldots, d_7$, $a_1, \ldots, a_{11}$ be its generators. Then T_0 is embedded isomorphically into the sublattice $\langle e_1, ..., e_6 \rangle \oplus \langle d_2, d_3 \rangle \subset L_{\text{root}}$. We calculate

$$W_{\text{root}} = \langle (d_1{+}d_2{+}2d_3{+}d_4), d_5, d_6, d_7 \rangle \oplus \langle a_1, ..., a_{11} \rangle.$$

By (NIEMEIER, 1973), $L/L_{\text{root}} = \mathbb{Z}/12\mathbb{Z}$, so there is a single element of order 2 modulo L_{root}. We can find μ in this class such that $2\mu \notin W_{\text{root}}$. Explicitly,

$$\mu = \frac{d_1 + d_2 + a_1 + a_3 + a_5 + a_7 + a_9 + a_{11}}{2}.$$

Consequently, the torsion subgroup is trivial. □

3.4.4 Classification with respect to automorphisms

In this section, we apply the results in Section 3.6 to the fibrations in Table 6.

Proposition 3.4.7. *The fibrations 1.1, 1.6, 1.7, 2.1, 2.3, 3.1, 4.1, 4.7, 4.8, 5.1, 5.2, 6.1, 6.2, 6.8, 6.9, 7.1, 8.1, 8.6, 9.1, 10.1, 10.5 in Table 6 are of type 1 with respect to σ.*

Demonstração. On all fibrations listed apart from 1.1, 1.7 and 4.8, the ADE-type T has a component of type E_6 or E_8. Thus, by Table 1, they have a fiber of type IV^* or II^*. The three remaining fibration have ADE-type $T = A_2^{\oplus \ell}$ for $\ell > 2$, so in particular they have more than 2 reducible fibers. By Proposition 3.3.12 and Corollary 3.3.13, all fibration listed cannot be of type 2. Since by assumption σ acts trivially on $\mathrm{NS}(X)$, the fibrations must be of type 1. □

Proposition 3.4.8. *The fibrations 1.2–1.5, 1.8–1.10, 2.2, 4.2–4.6, 4.9–4.11, 5.3, 6.3–6.7, 6.10–6.12, 7.2, 8.2–8.5, 8.7, 8.8, 9.2, 10.2, 10.3, 10.4, 10.6 in Table 6 are of type 2 with respect to σ.*

Demonstração. All listed fibrations apart from 9.2, 10.2 and 10.6 have positive rank, so by Proposition 3.2.12 they cannot be of type 1. The ADE-types T of the three remaining fibrations all have a component D_n for $n > 4$, so by Table 1 they have fibers of type I_{n-4}^*. Therefore, by Proposition 3.3.1, these fibrations are also not of type 1. □

Since we are able to determine the types of every fibration with respect to the automorphism σ, we can use our results on the reducible fibers on fibrations of type 1 and 2 to determine the following.

Corollary 3.4.9. *On all fibration on Table 6, the ADE-type T determines the Kodaira types of all reducible fibers.*

Demonstração. By Table 1, the type T_v of a reducible fiber F_v corresponds uniquely to the Kodaira-type of F_v, with the exception of A_1 and A_2. If $T_v = A_1$, then the fiber F_v can be of type I_2 or III; similarly if $T_v = A_2$ then F_v is of type I_3 or IV. By Propositions 3.3.1 and 3.3.12 and Corollary 3.3.13, the fibrations on Table 6 do not admit fibers of type I_2 or III. Furthermore, if T has a component of type A_2, then it corresponds to a fiber of Kodaira type IV if π is of type 1 and I_2 if π is of type 2 with respect to σ. □

3.5 Elliptic fibrations on X_3

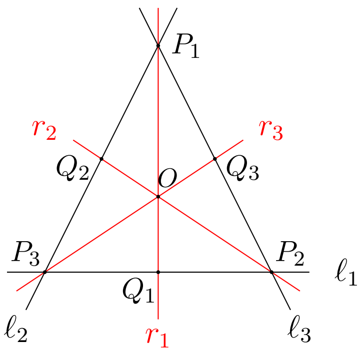
The X_3 surface is the minimal resolution of $(E_{\zeta_3} \times E_{\zeta_3})/\rho$, where ζ_3 is a primitive cube root of unity, E_{ζ_3} the elliptic curve with torus $\mathbb{C}/(\mathbb{Z} + \zeta_3\mathbb{Z})$ and ρ the automorphism given by $\rho(z_1, z_2) = (\zeta_3 z_1, \zeta_3^{-1} z_2)$. It was first studied by Shioda and Inose in (INOSE; SHIODA, 1977), and subsequently by Vinberg in (VINBERG, 1983), where it was first denoted by X_3 and described as one of the most algebraic K3 surfaces. The X_3 surface was also studied in (GEEMEN; TOP, 2006), where it was seen as a special fiber in a larger family of K3 surfaces. In particular, we know that $\rho(X_3) = 20$ and X_3 admits a non-symplectic automorphism σ of order 3 which acts trivially on $\mathrm{NS}(X_3)$. Therefore, by the classification in Table 5, we know that $\mathrm{NS}(X_3) = U \oplus E_8^{\oplus 2} \oplus A_2$ and the fixed locus of σ consists of 9 isolated points and 6 rational curves. Furthermore, by (ARTEBANI; SARTI, 2008, Proposition 5.1), the moduli space of K3 surfaces with these properties is irreducible. In particular, since $\rho(X_3) = 20$, any K3 surface X with $\rho(X) = 20$ and a non-symplectic automorphism of order 3 acting trivially on $\mathrm{NS}(X)$ is isomorphic to X_3.

The $\mathcal{J}_2$-classification of elliptic fibrations on X_3 appears on Table 6, but it was first presented as an application of the Kneser–Nishiyama method in (NISHIYAMA, 1996). Furthermore, (BRAUN; KIMURA; WATARI, 2013, Corollary D) shows that $\mathcal{J}_1(X_3) = \mathcal{J}_2(X_3)$, that is, there is exactly one elliptic fibration on X_3 in each $\mathcal{J}_2$-class modulo $\mathrm{Aut}(X_3)$.

In this section, we use the X_3 surface in order to show the relation between the elliptic fibrations on a K3 surface and the linear systems on the resolution of its quotient by a non-symplectic automorphism of prime order. This method also allows us to find explicit Weierstrass equations for an elliptic fibration in each class. We start in Section 3.5.1 by constructing X_3 as a base change of a rational elliptic surface R by a cubic Galois cover, which endows the K3 surface with a non-symplectic automorphism σ of order 3. In Section 3.5.2, we provide an elliptic fibration in each class of $\mathcal{J}_1(X)$ by describing the fiber class and a section, and apply Theorem 3.3.15 to classify each in relation to σ. In Section 3.5.3, we apply Propositions 3.3.17 and 3.3.16 to provide Weierstrass equation for each fibration.

3.5.1 Construction of X_3

In this section, we construct X_3 as the base change of a specific rational elliptic surface. Let R be constructed by the pencil of cubics $\Lambda = s\mathcal{F} + t\mathcal{G}$ in $\mathbb{P}^2$, where $\mathcal{F}$ is given by $xyz = 0$ and $\mathcal{G}$ by $(x-y)(y-z)(z-x) = 0$ (see Proposition 1.2.2).



$\ell_1 :$	$x = 0$	$P_1 = [1{:}0{:}0]$
$\ell_2 :$	$y = 0$	$P_2 = [0{:}1{:}0]$
$\ell_3 :$	$z = 0$	$P_3 = [0{:}0{:}1]$
$r_1 :$	$y - z = 0$	$Q_1 = [0{:}1{:}1]$
$r_2 :$	$z - x = 0$	$Q_2 = [1{:}0{:}1]$
$r_3 :$	$x - y = 0$	$Q_3 = [1{:}1{:}0]$
		$O = [1{:}1{:}1]$

Figura 3 – Cubics generating Λ

The base points of Λ lie on the scheme theoretic intersection $\mathcal{F} \cap \mathcal{G}$, and consist of $P_1, P_2, P_3, Q_1, \ldots, Q_6$. The points Q_4, Q_5, Q_6 are infinitely near to P_1, P_2, P_3, respectively, and correspond to the tangent directions of r_1, r_2, r_3. Blowing up the base points, we obtain the rational elliptic surface R, whose only reducible fibers are $F_a := \pi_R^{-1}([0{:}1])$ of type IV and $F_b := \pi_R^{-1}([1{:}0])$ of type I_6, i.e. the strict transforms of $\mathcal{F}$ and $\mathcal{G}$. The exceptional divisors $H_1, \ldots, H_6$ above $Q_1, \ldots, Q_6$ determine sections of π_R.

The curves in Figure 4 have self-intersections $\ell_i^2 = E_i^2 = r_i^2 = -2$, and $H_i^2 = -1$. By (OGUISO; SHIODA, 1991, Main Theorem), the Mordell–Weil group of π_R is $\mathrm{MW}(\pi_R) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

We apply the base change by the cubic Galois cover $\tau_{\mathbb{P}^1} \colon \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified at $[0{:}1]$ and $[1{:}0]$, obtaining a K3 surface with an elliptic fibration $\pi_X \colon X \to \mathbb{P}^1$ (see Proposition 3.6.5). The fibers above the ramification points of $\tau_{\mathbb{P}^1}$ are $F_a^X := \pi_X^{-1}([0{:}1])$ of type I_0, and $F_b^X := \pi_X^{-1}([1{:}0])$ of type I_{18}, and every other fiber is irreducible (see (MIRANDA, 1989, Table VI.4.1)).

Proposition 3.5.1. *Let σ be the automorphism of X induced by the base change. Then, σ acts trivially on* $\mathrm{NS}(X)$.

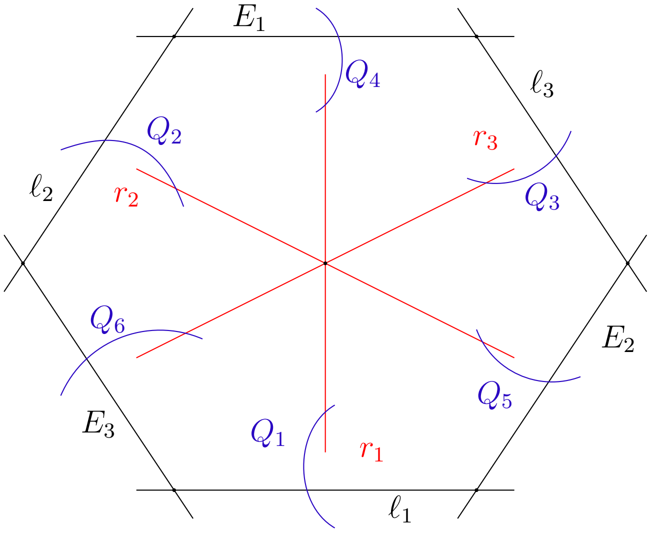


Figura 4 – Reducible fibers of π_R

Demonstração. By Proposition 3.6.12, we know that the generic fiber of π_X is given by $\mathcal{F} + t^3\mathcal{G}$. Furthermore, σ acts as $t \mapsto \zeta_3 t$. Thus, σ preserves every section and permutes the fibers of π_X, except for F_a^X and F_b^X. Since every other fiber in irreducible and σ fixes the fiber class, all that remains is describing the action of σ over F_a^X and F_b^X. We start by looking at the corresponding fibers in $R \times_{\mathbb{P}^1} \mathbb{P}^1$, which we denote by $\hat{F}_a$ and $\hat{F}_b$, respectively. These fibers are isomorphic to the corresponding fibers in π_R, and all their components are fixed by the induced automorphism. However, their singular points are also singularities of $R \times_{\mathbb{P}^1} \mathbb{P}^1$. In order to arrive at X and σ, we need to blow-up these singularities and lift the automorphism to the exceptional divisor.

Let p be a singular point of $\hat{F}_b$. Then, $R \times_{\mathbb{P}^1} \mathbb{P}^1$ is given locally by $t = xy$, with $(0,0,0)$ corresponding to p and $(x,y,t) \mapsto (x,y,\zeta_3 t)$ the Galois automorphism obtained by the base change. Blowing-up the origin, we have $t = x_1 y_1$, where $x_1 = \frac{x}{t}$ and $Y_1 = \frac{y}{t}$. The exceptional divisor at $t = 0$ consists of 2 rational curves given by $x_1 = 0$ and $y_1 = 0$. We can extend the automorphism as $(x_1, y_1, t) \mapsto (\zeta_3^2 c_1, \zeta_3^2 y_1, \zeta_3 t)$. Thus, this automorphism acts on each component of the exceptional divisor with order 3, and their intersection at $x_1 = y_1 = 0$ is an isolated fixed point.

Now, let p be the singular point of $\hat{F}_a$. In this case, the surface $R \times_{\mathbb{P}^1} \mathbb{P}^1$ is given locally by $t = xy(x+y)$. Blowing up the origin we have $t_2^3 = x_2^2 + x_2$, with $x_2 = \frac{x}{y}$ and $t_2 = \frac{t}{y}$. The exceptional divisor at $y = 0$ is an elliptic curve, and the action can be extended as $(x_2, y, t_2) \mapsto (x_2, y, \zeta_3 t_2)$. Thus, σ acts with order 3 on the exceptional divisor, and fixes 3 points given by $(0,0,0)$, $(1,0,0)$ and the point at infinity of the elliptic curve. These fixed points lie on the intersection with the strict transforms of the components $\hat{F}_a$. These components are (-1)-curves, and after their contraction we obtain X. With this, we are able to describe the action of σ on X, and see that σ acts trivially on $\mathrm{NS}(X)$. □

The description of the action of σ over F_a^X and F_b^X shows that the fixed locus of σ contains 6 rational curves, corresponding to the components of F_b^X coming from the I_6 fiber F_b, and 9 isolated fixed points, corresponding to the 3 fixed points on F_a^X and the 6 points on the intersection of the curves of F_b^X which are not on the fixed locus. Notice that this agrees with the description of the fixed locus on (ARTEBANI; SARTI, 2008, Proposition 3.2). Furthermore, we have the following corollary.

Corollary 3.5.2. *The surface X is isomorphic to X_3.*

Demonstração. By the Shioda–Tate formula, $\rho(X) = 20$, and by Proposition 3.5.1, σ is a non-symplectic automorphism of order 3 acting trivially on $\mathrm{NS}(X)$. Therefore, by (ARTEBANI; SARTI, 2008, Proposition 5.1), X is isomorphic to X_3. □

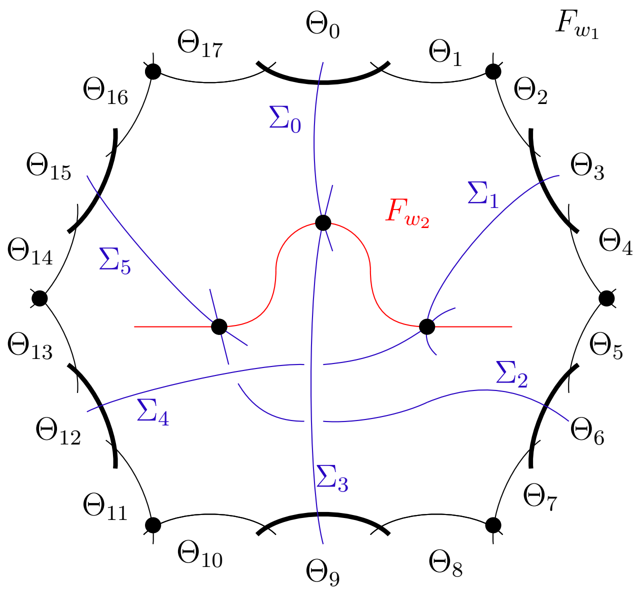


Figura 5 – Ramified fibers of π_X

Figure 5 describes the fibers F_a^X and F_b^X of π_X. The curves Θ_i are the components of F_b^X and Σ_i are the sections inherited from H_i in R. The self-intersections of the portrayed curves are $\Theta_i^2 = \Sigma_i^2 = -2$, $(F_a^X)^2 = 0$. The curves in the fixed locus of σ are highlighted as bold, and the isolated fixed points are marked by black dots.

3.5.2 Elliptic fibrations on X_3

By Proposition 1.3.13, elliptic fibrations on X are equivalent to embeddings $U \hookrightarrow \mathrm{NS}(X)$. Then, for classes $L, M \in \mathrm{NS}(X)$ such that $L^2 = 0$, $M^2 = -2$ and $L \cdot M = 1$, there is a fibration $\pi_{L,M} \colon X \to \mathbb{P}^1$ such that L is the fiber class, and M the class of the zero-section. Our goal is to create divisors L_i and M_i which induce an elliptic fibrations $\pi_i \colon X \to \mathbb{P}^1$ for each fibration 10.i in Table 6. We can create L_i, M_i using only Θ_j's and Σ_k's as components.

Proposition 3.5.3. *For $i = 1, ..., 6$, each pair L_i, M_i in Table 10 induces an elliptic fibration $\pi_i \colon X \to \mathbb{P}^1$ in the same $\mathcal{J}_1$-class as fibration 10.i in Table 6.*

No.	L_i	M_i
1	$2\Theta_{16} + 4\Theta_{17} + 6\Theta_0 + 3\Sigma_0 + 5\Theta_1 + 4\Theta_2 + 3\Theta_3 + 2\Theta_4 + \Theta_5$	Θ_6
2	$\Theta_{17} + \Sigma_0 + 2\sum_{j=0}^{12} \Theta_j + \Sigma_4 + \Theta_{13}$	Θ_{16}
3	$\Theta_{15} + 2\Theta_{16} + 3\Theta_{17} + 4\Theta_0 + 2\Sigma_0 + 3\Theta_1 + 2\Theta_2 + \Theta_3$	Θ_4
4	$\sum_{j=0}^{17} \Theta_j$	Σ_0
5	$3\Theta_0 + 2\Theta_1 + 2\Theta_{17} + 2\Sigma_0 + \Theta_2 + \Theta_{16} + \Sigma_3$	Θ_3
6	$\Theta_{17} + \Sigma_0 + 2\Theta_0 + 2\Theta_1 + 2\Theta_2 + 2\Theta_3 + \Sigma_1 + \Theta_4$	Θ_5

Tabela 10 – Divisors inducing elliptic fibrations on X_3

Demonstração. For each i, the divisor L_i lies in the fiber class of π_i. By construction, this divisor represents a reducible fiber of π_i, and its root type as a lattice must appear in the ADE-type of the fibration. For example, L_1 is a fiber of type II^*, which induces an E_8 in the ADE-type of π_1. The only fibration with this ADE-type possible in X by Table 6 is 10.1. Equivalently, L_2 induces a D_{16}, L_3 an E_7, L_4 an A_{17}, L_5 an E_6 and L_6 a D_7. These root lattices are all unique to the ADE-types of their respective fibrations. □

In order to apply Theorem 3.3.15, we need to describe $\tilde{X}$, given by the blow-up of the isolated fixed points of σ, and $\tilde{R}$, the quotient of $\tilde{X}$ by the lifting $\tilde{\sigma}$ of σ. The surface $\tilde{X}$ has 9 new components given by the exceptional curves of the blow-up $\eta \colon \tilde{X} \to X$. We denote the 6 curves above the intersection points of Θ_{3i+1} and Θ_{3i+2} by $\Theta_{3i+1,3i+2}$ (for $i = 0, ..., 5$), and the 3 curves above the intersection of Σ_i and Σ_{i+3} by $\Sigma_{i,i+3}$ (for $i = 0, 1, 2$). For each curve C in X, $\tilde{C}$ denotes the strict transform of C by η. The self intersections of each component are given by $\tilde{\Theta}_i^2 = -2$ if $i \equiv 0 \mod 3$, $\tilde{\Theta}_i^2 = -3$ if $i \not\equiv 0 \mod 3$, $\tilde{\Sigma}_i^2 = -3$, $\Theta_{i,i+1}^2 = \Sigma_{i,i+3}^2 = -1$ and $\tilde{F}_a^2 = -2$. The automorphism $\tilde{\sigma}$ of $\tilde{X}$ maintains the action of σ on each strict transform and fixes the exceptional curves.

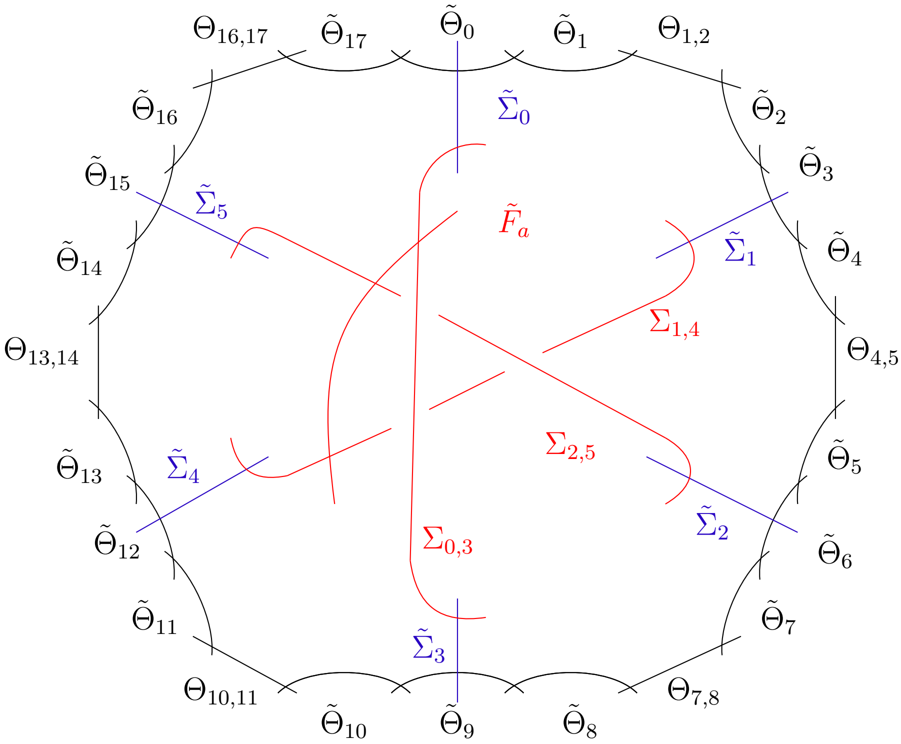


Figura 6 – Components of $\tilde{X}$

Let $\tilde{R} = \tilde{X}/\tilde{\sigma}$, and $\tilde{\tau}\colon \tilde{X} \to \tilde{R}$ the quotient map. For each component above, we calculate the pushforward by $\tilde{\tau}$ given as follows.

$$
\begin{array}{llll}
\tilde{\tau}_*\tilde{\Theta}_0 = \tilde{E}_1 & \tilde{\tau}_*\tilde{\Theta}_9 = \tilde{\ell}_1 & \tilde{\tau}_*\tilde{\Sigma}_0 = 3\tilde{H}_4 & \tilde{\tau}_*\Theta_{1,2} = R_1 \\
\tilde{\tau}_*\tilde{\Theta}_1 = 3S_1 & \tilde{\tau}_*\tilde{\Theta}_{10} = 3S_7 & \tilde{\tau}_*\tilde{\Sigma}_1 = 3\tilde{H}_3 & \tilde{\tau}_*\Theta_{4,5} = R_2 \\
\tilde{\tau}_*\tilde{\Theta}_2 = 3S_2 & \tilde{\tau}_*\tilde{\Theta}_{11} = 3S_8 & \tilde{\tau}_*\tilde{\Sigma}_2 = 3\tilde{H}_5 & \tilde{\tau}_*\Theta_{7,8} = R_3 \\
\tilde{\tau}_*\tilde{\Theta}_3 = \tilde{\ell}_3 & \tilde{\tau}_*\tilde{\Theta}_{12} = \tilde{E}_3 & \tilde{\tau}_*\tilde{\Sigma}_3 = 3\tilde{H}_1 & \tilde{\tau}_*\Theta_{10,11} = R_4 \\
\tilde{\tau}_*\tilde{\Theta}_4 = 3S_3 & \tilde{\tau}_*\tilde{\Theta}_{13} = 3S_9 & \tilde{\tau}_*\tilde{\Sigma}_4 = 3\tilde{H}_6 & \tilde{\tau}_*\Theta_{13,14} = R_5 \\
\tilde{\tau}_*\tilde{\Theta}_5 = 3S_4 & \tilde{\tau}_*\tilde{\Theta}_{14} = 3S_{10} & \tilde{\tau}_*\tilde{\Sigma}_5 = 3\tilde{H}_2 & \tilde{\tau}_*\Theta_{16,17} = R_6 \\
\tilde{\tau}_*\tilde{\Theta}_6 = \tilde{E}_2 & \tilde{\tau}_*\tilde{\Theta}_{15} = \tilde{\ell}_2 & \tilde{\tau}_*\Sigma_{0,3} = \tilde{r}_1 & \tilde{\tau}_*\tilde{F}_a = 3E_O \\
\tilde{\tau}_*\tilde{\Theta}_7 = 3S_5 & \tilde{\tau}_*\tilde{\Theta}_{16} = 3S_{11} & \tilde{\tau}_*\Sigma_{1,4} = \tilde{r}_3 & \\
\tilde{\tau}_*\tilde{\Theta}_8 = 3S_6 & \tilde{\tau}_*\tilde{\Theta}_{17} = 3S_{12} & \tilde{\tau}_*\Sigma_{2,5} = \tilde{r}_2 &
\end{array}
$$

The intersection pattern between distinct components is maintained, but the self-intersections are changed. The pushforwards of curves fixed by $\tilde{\sigma}$ have the self-intersection multiplied by 3. On the other hand, the pushforward of curves preserved by $\tilde{\sigma}$ (i.e., $\tilde{\sigma}$ acts on the curve with order 3) have the self intersection divided by 3. We obtain $\tilde{E}_i^2 = \tilde{\ell}_i^2 = -6$, $R_i^2 = \tilde{r}_i^2 = -3$, and $S_i^2 = \tilde{H}_i^2 = E_O^2 = -1$. Notice that even though $g(\tilde{F}_a) = 1$, since $\tilde{\sigma}$ fixes 3 distinct points in $\tilde{F}_a$, by Riemann–Hurwitz we have $g(E_O) = 0$.

If we contract the (-1)-curves $S_1, ..., S_{12}$ and E_O, and subsequently contract the curves $R_1, ..., R_6$, we obtain a birational morphism $\varepsilon \colon \tilde{R} \to R$, with $\varepsilon_* \tilde{E}_i = E_i, \varepsilon_* \tilde{\ell}_i = \ell_i, \varepsilon_* \tilde{H}_i = H_i$ and $\varepsilon_* \tilde{r}_i = r_i$. Since the canonical divisor of R is equal to $-F$ in $\mathrm{NS}(R)$, we can use ε to calculate the canonical divisor of $\tilde{R}$

$$K_{\tilde{R}} = \sum_{i=1}^{6} R_i + 2\sum_{i=1}^{12} S_i - \sum_{i=1}^{3} \tilde{r}_i - 2E_O.$$

Theorem 3.5.4. *Let $\pi_1, \ldots, \pi_6$ be the elliptic fibrations on (X, σ) presented in Proposition 3.5.3. Then, π_1 and π_5 are of type 1 and $\pi_2, \pi_3, \pi_4, \pi_6$ are of type 2, with respect to σ.*

Demonstração. For every L_i in Table 10, we first apply the pullback by η, then the pushforward by $\tilde{\tau}$. The divisor $\tilde{\tau}_*(\eta^* L_i)$ induces a linear system in $\tilde{R}$, which by Theorem 3.3.15 can be used to classify π_i with respect to the action of σ. If π_i is of type 1, then $\tilde{\tau}_*(\eta^* L_i) = 3\tilde{L}_i$, where $\tilde{L}_i$ is a generalized conic bundle class in $\tilde{R}$, so $\tilde{\tau}_*(\eta^* L_i) \cdot K_{\tilde{R}} = 3\tilde{L}_i \cdot K_{\tilde{R}} = 3 \cdot (-2) = -6$. If π_i is of type 2, then $\tilde{\tau}_*(\eta^* L_i) = \tilde{L}_i$ is a splitting genus 1 pencil, and $\tilde{L}_i \cdot K_{\tilde{R}} = 0$. Since σ acts trivially on $\mathrm{NS}(X)$, there are no fibrations of type 3. The explicit calculations of $\tilde{\tau}_*(\eta^* L_i)$ and its intersection with $K_{\tilde{R}}$ are presented in Table 11. □

No.	$\tilde{\tau}_*(\eta^* L_i)$	$\tilde{\tau}_*(\eta^* L_i) \cdot K_{\tilde{R}}$
1	$3\tilde{L}_1 = 3(2S_{11} + 2R_6 + 4S_{12} + 2\tilde{E}_1 + 3\tilde{H}_4 + \tilde{r}_1 + 5S_1 + 3R_1 + 4S_2 + \tilde{\ell}_3 + 2S_3 + R_2 + S_4)$	-6
2	$\tilde{L}_2 = R_6 + 3S_{12} + \tilde{r}_1 + 3\tilde{H}_4 + 2\tilde{E}_1 + 6S_1 + 4R_1 + 6S_2 + 2\tilde{\ell}_3 + 6S_3 + 4R_2 + 6S_4 + 2\tilde{E}_2 + 6S_5 + 4R_3 + 6S_6 + 2\tilde{\ell}_1 + 6S_7 + 4R_4 + 6S_8 + 2\tilde{E}_3 + 3\tilde{H}_6 + \tilde{r}_3 + 3S_9 + R_5$	0
3	$\tilde{L}_3 = \tilde{\ell}_2 + 6S_{11} + 5R_6 + 9S_{12} + 4\tilde{E}_1 + 6\tilde{H}_4 + 2\tilde{r}_1 + 9S_1 + 5R_1 + 6S_2 + \tilde{\ell}_3$	0
4	$\tilde{L}_4 = \sum_{i=1}^{3} \tilde{E}_i + \sum_{i=1}^{3} \tilde{\ell}_i + 3\sum_{i=1}^{12} S_i + 2\sum_{i=1}^{6} \tilde{R}_i$	0
5	$3\tilde{L}_5 = 3(\tilde{E}_1 + 2S_1 + R_1 + S_2 + 2S_{12} + R_6 + S_{11} + 2\tilde{H}_4 + \tilde{r}_1 + \tilde{H}_1)$	-6
6	$\tilde{L}_6 = R_6 + 3S_{12} + \tilde{r}_1 + 3\tilde{H}_4 + 2\tilde{E}_1 + 6S_1 + 4R_1 + 6S_2 + 2\tilde{\ell}_3 + 3\tilde{H}_3 + \tilde{r}_3 + 3S_3 + R_2$	0

Tabela 11 – Divisors induced in $\tilde{R}$

3.5.3 Weierstrass Equations of the elliptic fibrations

In this section, we apply the method described in Section 3.3.4 to find Weierstrass equations for each π_i in Table 10. Let Γ_i be the pencils of curves in $\mathbb{P}^2$ induced by each π_i. We describe the geometry of Γ_i by applying the following substitutions for each $\tilde{L}_i$ in Table 11, where $\tilde{h}$ is the pullback of the line class $h \in \text{Pic}(\mathbb{P}^2)$ to $\text{Pic}(\tilde{R})$.

$$\begin{aligned}
\tilde{l_1} &= \tilde{h} - \tilde{E_2} - \tilde{E_3} - \tilde{H_1} - \tilde{H_5} - \tilde{H_6} - R_2 - 2R_3 - 2R_4 - R_5 \\
&\quad - S_3 - 2S_4 - 3S_5 - 3S_6 - 3S_7 - 3S_8 - 2S_9 - S_{10}, \\
\tilde{l_2} &= \tilde{h} - \tilde{E_1} - \tilde{E_3} - \tilde{H_2} - \tilde{H_4} - \tilde{H_6} - R_1 - R_4 - 2R_5 - 2R_6 \\
&\quad - 2S_1 - S_2 - S_7 - 2S_8 - 3S_9 - 3S_{10} - 3S_{11} - 3S_{12}, \\
\tilde{l_3} &= \tilde{h} - \tilde{E_1} - \tilde{E_2} - \tilde{H_3} - \tilde{H_4} - \tilde{H_5} - 2R_1 - 2R_2 - R_3 - R_6 \\
&\quad - 3S_1 - 3S_2 - 3S_3 - 3S_4 - 2S_5 - S_6 - S_{11} - 2S_{12}, \\
\tilde{r_1} &= \tilde{h} - E_O - \tilde{E_1} - \tilde{H_1} - 2\tilde{H_4} - R_1 - R_6 - 2S_1 - S_2 - S_{11} - 2S_{12}, \\
\tilde{r_2} &= \tilde{h} - E_O - \tilde{E_2} - \tilde{H_2} - 2\tilde{H_5} - R_2 - R_3 - S_3 - 2S_4 - 2S_5 - S_6, \\
\tilde{r_3} &= \tilde{h} - E_O - \tilde{E_3} - \tilde{H_3} - 2\tilde{H_6} - R_4 - R_5 - S_7 - 2S_8 - 2S_9 - S_{10}.
\end{aligned}$$

As an example, applying these substitutions to $\tilde{L_1}$ we obtain $\tilde{L_1} = 2\tilde{h} - E_O - \tilde{E_2} - \tilde{H_1} - \tilde{H_3} - \tilde{H_5} - R_2 - R_3 - S_3 - 2S_4 - 2S_5 - S_6$. The component $2\tilde{h}$ indicates that the linear system Γ_1 in $\mathbb{P}^2$ is composed of conics. The negative components E_O, E_2, H_1 and H_3 indicate that the conics in Γ_1 pass through the points O, P_2, Q_1 and Q_3 in $\mathbb{P}^2$, and the remaining components come from the pullback of the former. Consequently, Γ_1 is the pencil of conics through O, P_2, Q_1, Q_3. We can do the same for the rest of the divisors $\tilde{L_i}$ (see Table 12).

The description of these systems allows us to use Propositions 3.3.17 and 3.3.16 and find Weierstrass equations for each fibration in Proposition 10.

Proposition 3.5.5. *The elliptic fibrations π_1 and π_5 are given by the following equations in Weierstrass form:*

$$\begin{aligned}
\pi_1 &: y^2 - \tfrac{2v^4 - 4v^3}{\alpha(v)} y = x^3 + \tfrac{v^5}{\beta(v)}, \\
\pi_5 &: y^2 - \tfrac{v^2 - 2v + 1}{v} y = x^3,
\end{aligned}$$

No.	$\tilde{L}_i$	Γ_i
1	$\tilde{L}_1 = 2\tilde{h} - E_O - \tilde{E}_2 - \tilde{H}_1 - \tilde{H}_3 - \tilde{H}_5 - R_2 - R_3 - S_3 - 2S_4 - 2S_5 - S_6$	Conics through P_2, Q_1, Q_3, O.
2	$\tilde{L}_2 = 6\tilde{h} - 2E_O - \tilde{E}_1 - 2\tilde{E}_2 - \tilde{E}_3 - 3\tilde{H}_1 - 3\tilde{H}_3 - \tilde{H}_4 - 4\tilde{H}_5 - \tilde{H}_6 - R_1 - 2R_2 - 2R_3 - R_4 - 2R_5 - 2R_6 - 2S_1 - S_2 - 2S_3 - 4S_4 - 4S_5 - 2S_6 - S_7 - 2S_8 - 3S_9 - 3S_{10} - 3S_{11} - 3S_{12}$	Sextics through Q_1, Q_3 (multiplicity 3), P_2 (tacnode tangent to r_2), O (multiplicity 2), P_1, P_3 (tangent to l_2).
3	$\tilde{L}_3 = 4\tilde{h} - 2E_O - \tilde{E}_2 - \tilde{E}_3 - 2\tilde{H}_1 - \tilde{H}_2 - \tilde{H}_3 - \tilde{H}_5 - \tilde{H}_6 - 2R_2 - R_3 - R_4 - 2R_5 - 3S_3 - 3S_4 - 2S_5 - S_6 - S_7 - 2S_8 - 3S_9 - 3S_{10}$	Quartics through O, Q_1 (multiplicity 2), P_2, P_3, Q_2, Q_3 (multiplicity 1).
4	$\tilde{L}_4 = 3\tilde{h} - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{H}_1 - \tilde{H}_2 - \tilde{H}_3 - 2\tilde{H}_4 - 2\tilde{H}_5 - 2\tilde{H}_6 - R_1 - R_2 - R_3 - R_4 - R_5 - R_6 - 2S_1 - S_2 - S_3 - 2S_4 - 2S_5 - S_6 - S_7 - 2S_8 - 2S_9 - S_{10} - S_{11} - 2S_{12}$	Cubics through Q_1, Q_2, Q_3 P_1 (tangent to r_1), P_2 (tangent to r_2), P_3 (tangent to r_3).
5	$\tilde{L}_5 = \tilde{h} - E_O$	Lines through O.
6	$\tilde{L}_6 = 4\tilde{h} - 2E_O - \tilde{E}_1 - 2\tilde{E}_2 - \tilde{E}_3 - \tilde{H}_1 - \tilde{H}_4 - 2\tilde{H}_5 - 2\tilde{H}_6 - R_1 - 2R_2 - 2R_3 - R_4 - R_5 - 2R_6 - 2S_1 - S_2 - 3S_3 - 6S_4 - 4S_5 - 2S_6 - S_7 - 2S_8 - 2S_9 - S_{10} - 2S_{11} - S_{12}$	Quartics through O, P_2 (multiplicity 2), P_1, Q_1, (multiplicity 1), P_3 (tangent to r_3).

Tabela 12 – Linear systems Γ_i

where,

$$\alpha(v) = v^{13}-16v^{12}+120v^{11}-554v^{10}+1742v^{9}-3903v^{8}+6337v^{7} -7435v^{6}+6171v^{5}-3470v^{4}+1229v^{3}-246v^{2}+25v-1,$$

$$\beta(v) = v^{22}-27v^{21}+351v^{20}-2915v^{19}+17310v^{18}-77975v^{17}+275920v^{16}-783765v^{15}+1811095v^{14} -3429800v^{13}+5338045v^{12}-6819500v^{11}+7115140v^{10}-6008210v^{9}+4051240v^{8}-2141208v^{7} +865711v^{6}-259643v^{5}+55665v^{4}-8165v^{3}+771v^{2}-42v+1.$$

Demonstração. By Proposition 3.3.16, we need to calculate the restriction of $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$ to each $D_{i,v}$ in Λ_i, which we write as $f_{i,v}\colon D_{i,v} \to \mathbb{P}^1$. To do this, we determine, for each $C_{i,v} \subset \mathbb{P}^2$ induced by $D_{i,v}$ in Γ_i, a map $\rho_{i,v}\colon \mathbb{P}^1 \to \mathbb{P}^2$ such that $\rho_{i,v}(\mathbb{P}^1) = C_{i,v}$. We can write these maps as follows.

$$\rho_{1,v}([u_1{:}u_2]) = [u_1^2 - vu_1u_2 : u_1^2 - vu_2^2 : u_1u_2 - vu_2^2],$$
$$\rho_{5,v}([u_1{:}u_2]) = [u_1 - vu_2 : u_1 - vu_1 : u_2 - vu_2].$$

By the construction of R as a pencil of cubics, the elliptic fibration $\pi_{\tilde{R}}$ is equivalent in an open set of $\tilde{R}$ to blowing down to $\mathbb{P}^2$ and composing with the rational map $\varphi([x{:}y{:}z]) = [xyz : (x-y)(y-z)(z-x)]$. Composing each $\rho_{i,v}$ with φ, we obtain maps $\varphi \circ \rho_{i,v} \colon \mathbb{P}^1 \cong C_{i,v} \dashrightarrow \mathbb{P}^1$, which we write as

$$\varphi \circ \rho_{5,v}([u_1{:}u_2]) = [(u_1 - vu_2)^2(u_1^2 - vu_2^2)u_1u_2 : v(u_1 - u_2)^3(u_1 - vu_2)u_1u_2],$$
$$\varphi \circ \rho_{1,v}([u_1{:}u_2]) = [(v-1)(u_1 - vu_2)u_1u_2 : v(u_1 - u_2)^3].$$

The maps $f_{1,v}$ and $f_{5,v}$ are the resolutions of indeterminacy of the maps above. The coordinates in $\varphi \circ \rho_{5,v}$ have no common factors, so $f_{5,v} = \varphi \circ \rho_{5,v}$. On the other hand, the coordinates of $\varphi \circ \rho_{1,v}$ have a common factor of $(u_1 - vu_2)u_1u_2$, so we have

$$f_{1,v}([u_1{:}u_2]) = [(u_1 - vu_2)(u_1^2 - vu_2^2) : v(u_1 - u_2)^3].$$

The equations for the fibrations are given by $f_{i,v}([u_1{:}u_2]) = \tau_{\mathbb{P}^1}([s{:}t])$ in $\mathbb{P}^1 \times \mathbb{P}^1$ over $k(v)$, thus we have

$$\pi_1 \colon s^3 v(u_1 - u_2)^3 = t^3(u_1 - vu_2)(u_1^2 - vu_2^2),$$
$$\pi_5 \colon s^3 v(u_1 - u_2)^3 = (v-1)t^3(u_1 - vu_2)u_1u_2.$$

Notice that the both equations admit a $k(v)$-point, namely $([v{:}1], [0{:}1])$ for π_1 and $([1{:}0], [0{:}1])$ for π_5. Thus, we can transform both equations to Weierstrass form, obtaining the result. ☐

Proposition 3.5.6. *The elliptic fibrations π_2, π_3 and π_6 are given by the following equations in Weierstrass form:*

$$\pi_2 \colon y^2 + 2xy = x^3 - \tfrac{3t^3+1}{t^3}x^2 + x,$$
$$\pi_3 \colon y^2 = x^3 + t^3x^2 - t^3x,$$
$$\pi_6 \colon y^2 - t^3xy + t^6y = x^3 - t^3x^2.$$

Demonstração. By Proposition 3.3.17, for each π_i, we need to find $F_{i,a}$ and $F_{i,b}$ the fibers above the ramification points of the base change, and then calculate the curves $\mathcal{C}_{i,a}$ and $\mathcal{C}_{i,b}$ they induce in Γ_i. Notice that the ADE-types of each fibration determines the Kodaira types of the ramified fibers, as noted in Remark 3.3.18. We can take $F_{i,a}$ to be the divisor L_i (see Table 10). Then, $F_{i,b}$ must be disjointed from $F_{i,a}$ and the same Kodaira type of the ramified fiber. For each π_i, we can choose as follows:

$$F_{2,b} = \Theta_{15} + \Sigma_5 + \Sigma',$$
$$F_{3,b} = \Theta_5 + \Sigma_2 + 2(\textstyle\sum_{i=6}^{12} \Theta_i) + \Theta_{13} + \Sigma_4,$$
$$F_{6,b} = \Sigma_2 + \textstyle\sum_{i=6}^{15} \Theta_i + \Sigma_5.$$

Here, Σ' is a section of π_X intersecting F_b^X in the component Θ_{15} and F_a^X in the same point as Σ_5. Next, we calculate $\tilde{\tau}_*(\eta^* F_{i,a})$ and $\tilde{\tau}_*(\eta^* F_{i,b})$ in $\mathrm{Pic}(\tilde{R})$. The components coming from curves in $\mathbb{P}^2$ will determine $\mathcal{C}_{i,a}$ and $\mathcal{C}_{i,b}$. For example, $\tilde{\tau}_*(\eta^* F_{3,a}) = \tilde{L}_3 = \tilde{\ell}_2 + 2\tilde{r}_1 + \tilde{\ell}_3 + (4\tilde{E}_1 + 6\tilde{H}_4 + 9R_1 + 5R_6 + 9S_1 + 6S_2 + 6S_2 + 6S_{11} + 9S_{12})$. Then, the induced curve $\mathcal{F}_3$ is given by $\ell_2 \cdot r_1^2 \cdot \ell_3^2 = x(y-z)^2 z = 0$.

This process is straightforward for every $F_{i,a}$, as well as for $F_{3,b}$ and $F_{6,b}$. For $F_{2,b}$, we first have to calculate $\tilde{\tau}_*(\eta^*(\Sigma'))$. Since Σ' passes through the same fixed point of σ as Σ_5, it follows that $\eta^*(\Sigma') = \tilde{\Sigma}' + \Sigma_{2,5}$, and $\tilde{\Sigma}'^2 = -3$. As $\tilde{\sigma}$ preserves $\tilde{\Sigma}'$, we calculate $\tilde{\tau}_*(\eta^*(\Sigma')) = 3H' + \tilde{r}_2$, where $H' = \tilde{\tau}(\tilde{\Sigma}')$ corresponds with the section of $\tilde{R}$ coming from the line $x + z = y$ in $\mathbb{P}^2$ passing through Q_1 and Q_2, and $H'^2 = -1$. Consequently, we have $\tilde{\tau}_*(\eta^* F_{2,b}) = \tilde{\ell}_2 + 3\tilde{H}_2 + 2\tilde{r}_2 + 3H'$.

We arrive at the following equations for π_2, π_3 and π_6 :

$$\begin{aligned}
\pi_2 &: \ x^2 z^2 (x-y)(y-z) + t^3 y (z-x)^2 (x-y+z)^3 = 0, \\
\pi_3 &: \ xy(x-y)^2 + t^3 z^2 (z-x)(y-z) = 0, \\
\pi_6 &: \ xz(y-z)^2 + t^3 y^2 (z-x)(x-y) = 0.
\end{aligned}$$

All three equations admit a $k(t)$-point, namely [0:0:1] for π_2, [1:0:0] for π_3 and [0:0:1] for π_6. Transforming the equations to Weierstrass form, we obtain the result. □

Remark 3.5.7. The action of σ can be given explicitly in the Weierstrass equations in the previous propositions. For the equations in Proposition 3.5.6, σ is given by $(x, y, t) \mapsto (x, y, \zeta_3 t)$, and for those in Proposition 3.5.5, σ is $(x, y, v) \mapsto (\zeta_3 x, y, v)$.

3.6 Classification of fibrations with respect to a non-symplectic automorphism

In this section, we adapt results from Section 3.3 to K3 surfaces X with a non-symplectic automorphism σ of order $p > 3$. Recall that by Theorem 1.3.8, p is at most equal to 19. We work under the following further assumption.

Assumption 3.6.1. We assume that the automorphism σ acts trivially on $\mathrm{NS}(X)$.

Under this assumption, by Remark 3.2.11, X does not admit any elliptic fibrations of type 3 with respect to σ. Furthermore, by Proposition 3.3.1, (X, σ) also does not admit fibrations of type 1. Consequently, in this section we deal exclusively with elliptic fibrations of type 2.

3.6.1 Base changes of rational elliptic surfaces

In this section, we perform base changes of rational elliptic surfaces to construct pairs (X, σ), where X is a K3 surface and σ a non-symplectic automorphism of prime order $p > 3$. Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface and $\pi_X\colon X \to \mathbb{P}^1$ the elliptic fibration obtained after the base change by a Galois covering $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree p. Take $v, v' \in \mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}(v') = v$. We denote their ramification index by $r(v'|v)$ and their fibers by $F_v = \pi^{-1}(v)$ and $F^X_{v'} = \pi_X^{-1}(v')$.

We want to know when X is a K3 surface. Since by construction X has an elliptic fibration with basis $\mathbb{P}^1$, by Proposition 1.3.14 this is true if and only if $e(X) = 24$. We use the following definition.

Definition 3.6.2. We define the function $\mathcal{C}(v'|v)$ as follows.

$$\mathcal{C}(v'|v) := r(v'|v) \cdot e(F_v) - e(F^X_{v'}).$$

By the Riemann–Hurwitz Theorem, $\tau_{\mathbb{P}^1}$ is ramified in 2 distinct points $a, b \in \mathbb{P}^1$. Take a', b' such that $\tau_{\mathbb{P}^1}(a') = a$ and $\tau_{\mathbb{P}^1}(b') = b$.

Lemma 3.6.3. *The Euler number of X is given by*

$$e(X) = 12p - \mathcal{C}(a'|a) - \mathcal{C}(b'|b).$$

Demonstração. By Proposition 1.1.13, we can write

$$e(X) = \sum_{v \in \mathbb{P}^1} e(F^X_v) = \sum_{v \in \mathbb{P}^1} \sum_{\tau_{\mathbb{P}^1}(v') = v} e(F^X_{v'}). \tag{3.1}$$

By Definition 3.6.2,

$$e(F^X_{v'}) = r(v'|v) \cdot e(F_v) - \mathcal{C}(v'|v). \tag{3.2}$$

Substituting Equation 3.2 into Equation 3.1, we obtain

$$e(X) = \sum_{v \in \mathbb{P}^1} e(F_v) \left(\sum_{\tau_{\mathbb{P}^1}(v') = v} r(v'|v) \right) - \sum_{v \in \mathbb{P}^1} \sum_{\tau_{\mathbb{P}^1}(v') = v} \mathcal{C}(v'|v) \tag{3.3}$$

Since $\tau_{\mathbb{P}^1}$ is a covering of degree p, we know $\sum_{\tau_{\mathbb{P}^1}(v') = v} r(v'|v) = p$. Applying Propositions 1.1.13 and 1.2.3, we know that $\sum_{v \in \mathbb{P}^1} e(F_v) = 12$. Furthermore, for $v' \neq a', b'$, we have $r(v'|v) = 1$ and by Proposition 1.1.23 $e(F^X_{v'}) = e(F_v)$. Thus, $\mathcal{C}(v'|v) = 0$, and we obtain the result by substituting in Equation 3.3. □

In the following proposition, we present a formula for $\mathcal{C}(v'|v)$ in terms of the Kodaira type of F_v and the ramification index $r(v'|v)$.

F_v	$\mathcal{C}(v'\|v)$	F_v	$\mathcal{C}(v'\|v)$
I_n	0	I_n^*	$12\left\lfloor \frac{m}{2} \right\rfloor$
II	$12\left\lfloor \frac{m}{6} \right\rfloor$	II^*	$12\left(m - \left\lfloor \frac{m}{6} \right\rfloor - \left\lceil \frac{m - 6\lfloor \frac{m}{6} \rfloor}{6} \right\rceil\right)$
III	$12\left\lfloor \frac{m}{4} \right\rfloor$	III^*	$12\left(m - \left\lfloor \frac{m}{4} \right\rfloor - \left\lceil \frac{m - 4\lfloor \frac{m}{4} \rfloor}{4} \right\rceil\right)$
IV	$12\left\lfloor \frac{m}{3} \right\rfloor$	IV^*	$12\left(m - \left\lfloor \frac{m}{3} \right\rfloor - \left\lceil \frac{m - 3\lfloor \frac{m}{3} \rfloor}{3} \right\rceil\right)$

Tabela 13 – Values of $\mathcal{C}(v'|v)$

Proposition 3.6.4. *Let $r(v'|v) = m$. The value of $\mathcal{C}(v'|v)$ is determined by the formulas in Table 13.*

Demonstração. Notice that the Kodaira type of $F_{v'}^X$ is known by 1.1.23, so we just need to check if $me(F_v) - \mathcal{C}(v'|v) = e(F_{v'}^X)$ for each possible Kodaira type of F_v. We present the proof for F_v of type IV or IV^*; the other cases are analogous.

Write $m = 3\left\lfloor \frac{m}{3} \right\rfloor + r$, with $r = 0$, 1 or 2. If F_v is of type IV, then $e(F_v) = 4$, and we have

$$\begin{aligned} me(F_v) - \mathcal{C}(v'|v) &= 4(3\left\lfloor \frac{m}{3} \right\rfloor + r) - 12\left\lfloor \frac{m}{3} \right\rfloor \\ &= 12\left\lfloor \frac{m}{3} \right\rfloor + 4r - 12\left\lfloor \frac{m}{3} \right\rfloor \\ &= 4r = \begin{cases} 0 & \text{if } m \equiv 0 \mod 3 \\ 4 & \text{if } m \equiv 1 \mod 3 \\ 8 & \text{if } m \equiv 2 \mod 3. \end{cases} \end{aligned}$$

By Proposition 1.1.23, $F_{v'}^X$ is of type I_0 if $m \equiv 0 \mod 3$, IV if $m \equiv 1 \mod 3$ and IV^* if $m \equiv 2 \mod 3$. Thus, the result is compatible with the values of $e(F_{v'}^X)$.

Write $r = m - 3\left\lfloor \frac{m}{3} \right\rfloor$. We have

$$\left\lceil \frac{m - 3\lfloor \frac{m}{3} \rfloor}{3} \right\rceil = \left\lceil \frac{r}{3} \right\rceil = \begin{cases} 0 & \text{if } m \equiv 0 \mod 3 \\ 1 & \text{otherwise.} \end{cases}$$

If F_v is of type IV^*, then $e(F_v) = 8$, and we write

$$
\begin{aligned}
&me(F_v) - \mathcal{C}(v'|v) = \\
&8m - 12\left(m - \left\lfloor \frac{m}{3} \right\rfloor - \left\lceil \frac{m - 3\lfloor \frac{m}{3} \rfloor}{3} \right\rceil\right) = \\
&12\left\lfloor \frac{m}{3} \right\rfloor + 12\left\lceil \frac{r}{3} \right\rceil - 4m = \\
&12\left\lfloor \frac{m}{3} \right\rfloor + 12\left\lceil \frac{r}{3} \right\rceil - 4(3\left\lfloor \frac{m}{3} \right\rfloor + r) = \\
&12\left\lceil \frac{r}{3} \right\rceil - 4r = \begin{cases} 0 & \text{if } m \equiv 0 \mod 3 \\ 12 - 4 = 8 & \text{if } m \equiv 1 \mod 3 \\ 12 - 8 = 4 & \text{if } m \equiv 2 \mod 3. \end{cases}
\end{aligned}
$$

This result is also compatible with the expected values for $e(F'_{v'})$. □

Proposition 3.6.5. *Let $\pi\colon R \to \mathbb{P}^1$ be a rational elliptic surface and $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ a Galois cover of prime degree p ramified at $a, b \in \mathbb{P}^1$. Let $\pi_X\colon X \to \mathbb{P}^1$ be the base change of $\pi\colon R \to \mathbb{P}^1$ by $\tau_{\mathbb{P}^1}$. Then, X is a K3 surface if and only if the fibers F_a and F_b have the Kodaira types described in Table 14, up to a permutation.*

p	F_a	F_b
5	III^*, IV^*	I_n, II
	I_n^*	III, IV
7	II^*, III^*	I_n
	IV^*	II, III
	I_n^*	IV
11	II^*	I_n
	III^*	II
	IV^*	III
13	III^*	II
17	IV^*	III
19	III^*	II

Tabela 14 – Ramified fibers when X is a K3 surface

Demonstração. By Proposition 1.3.14, X is K3 if and only if $e(X) = 24$, and by Lemma 3.6.3, $e(X) = 12p - \mathcal{C}(a'|a) - \mathcal{C}(b'|b)$. Thus, X is a K3 surface if and only if $\mathcal{C}(a'|a) + \mathcal{C}(b'|b) = 12(p-2)$. We can calculate $\mathcal{C}(a'|a) + \mathcal{C}(b'|b)$ using Proposition 3.6.4. We note that if $p = 13$ and F_a and F_b both of type IV^*, then $\mathcal{C}(a'|a) + \mathcal{C}(b'|b) = 132 = 12 \cdot 11$, but this is not possible in rational elliptic surfaces by Corollary 1.2.4. This brings us to the result. □

Remark 3.6.6. Let X be a K3 surface and $\pi_X\colon X \to \mathbb{P}^1$ be the base change of a rational elliptic surface by a Galois map $\tau_{\mathbb{P}^1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of prime degree p. The map $\tau_{\mathbb{P}^1}$ induces an automorphism $\sigma_\pi \in \mathrm{Aut}(\mathbb{P}^1)$, which can in turn be lifted to an automorphism $\sigma \in \mathrm{Aut}(X)$. The quotient of X by σ is a rational surface, so by Theorem 1.3.12, σ is non-symplectic. By construction, π_X is an elliptic fibration of type 2 on (X, σ). Notice that in general (X, σ) does not satisfy Assumption 3.6.1, i.e. σ may act non-trivially on $\mathrm{NS}(X)$. This always happens when π_X has an irreducible fiber distinct from the ramified fibers $F^X_{a'}$, $F^X_{b'}$.

3.6.2 Elliptic fibrations on (X, σ)

Let X be a K3 surface and $\sigma \in \mathrm{Aut}(X)$ a non-symplectic automorphism of order $p > 3$ under Assumption 3.6.1. In what follows, we use Proposition 3.6.5 to describe the configuration of fibers of an elliptic fibration $\pi\colon X \to \mathbb{P}^1$. Recall that π is of type 2 with respect to σ.

By Theorem 1.3.12, the quotient X/σ is rational, but, as in the case of $p = 3$, in general it is not smooth. By Proposition 1.3.10, the local action of σ around x can be linearized as $A_{p,t}$, and a fixed point $x \in X$ is isolated if and only if $t \geq 0$. In this case, the point $\tau(x) \in X/\sigma$ is a singularity of type $\frac{1}{p}(1, b)$ for some b such that $0 < b < p$ (see (REID, 2003)). Let $\varphi\colon \tilde{R} \to X/\sigma$ be the resolution of all the singularities of X/σ.

Proposition 3.6.7. *Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ). Then, π induces an elliptic fibration $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$.*

Demonstração. Let σ_π be the automorphism of $\mathbb{P}^1$ induced by σ and π. Choose $v_1 \in \mathbb{P}^1$ such that v_1 is not fixed by σ_π and its respective fiber F_{v_1} is smooth. Then, the orbit of F_{v_1} by σ consists of p distinct smooth fibers denoted by F_{v_i} for $i = 1, \ldots, p$, where $v_i = \sigma_\pi^{i-1}(v_1)$. In particular, none of the fibers F_{v_i} contain an isolated fixed point of σ.

Let $D = \tau(F_{v_1}) \subset X/\sigma$. Then, σ defines a unramified cover of degree p of D given by p disjointed smooth curves of genus 1. Thus D is a smooth curve and by the Riemann–Hurwitz Theorem it also has genus 1. Furthermore, $D^2 = \tau_*(F_{v_1}) \cdot D = F_{v_1} \cdot \tau^*(D) = F_{v_1} \cdot (F_{v_1} + \ldots + F_{v_p}) = 0$. Finally, let $\tilde{D} = \varphi^{-1}(D) \subset \tilde{R}$. Notice that D does not contain the singularities resolved by φ, since the isolated fixed points of σ are not in F_{v_i}. Thus, $\tilde{D}$ is also a smooth genus 1 curve with $\tilde{D}^2 = 0$, and the linear system $|D|$ induces a fibration of genus 1 curves $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$.

It remains to see that $\pi_{\tilde{R}}$ admits a section. Let Σ_0 be the zero-section of π. Then, let $C = \tau(\Sigma_0)$ and $\tilde{C}$ be the strict transform of C by φ. We can calculate $D \cdot C = \tau_*(F_{v_1}) \cdot C = F_{v_1} \cdot \tau^*(C) = F_{v_1} \cdot \Sigma_0 = 1$. Since $D \cap C$ is not in the center of the resolution φ, $\tilde{D} \cdot \tilde{C} = 1$. Therefore, $\pi_{\tilde{R}} \colon \tilde{R} \to \mathbb{P}^1$ is an elliptic fibration with $\tilde{D}$ as its fiber class and $\tilde{C}$ as its zero-section. □

Assume $\pi_{\tilde{R}} \colon \tilde{R} \to \mathbb{P}^1$ is not relatively minimal. Then, there is a birational morphism $\eta_R \colon \tilde{R} \to R$ such that R is a smooth rational surface endowed with a relatively minimal elliptic fibration $\pi_R \colon R \to \mathbb{P}^1$ such that $\pi_{\tilde{R}} = \eta_R \circ \pi_R$.

Proposition 3.6.8. *Let $\pi \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ). Then, there is a map $\tau_\pi \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that π is the base change of $\pi_R \colon R \to \mathbb{P}^1$ by τ_π, and σ is the induced automorphism.*

Demonstração. Let σ_π be the automorphism of $\mathbb{P}^1$ induced by σ and π and $\tau_\pi \colon \mathbb{P}^1 \to \mathbb{P}^1$ the quotient of $\mathbb{P}^1$ by the action of σ_π.

Let a', b' be the ramification points of τ_π, and $\tau_\pi(a') = a, \tau_\pi(b') = b$. Let $U = \pi^{-1}(\mathbb{P}^1 \setminus \{a', b'\}) \subset X$, $\tilde{V} = \pi_{\tilde{R}}^{-1}(\mathbb{P}^1 \setminus \{a, b\})$ and $V = \pi_R^{-1}(\mathbb{P}^1 \setminus \{a, b\})$. Since the isolated fixed points of σ lie on the fibers $F_{a'}^X, F_{b'}^X$ of π, the exceptional curves of $\varphi \colon \tilde{R} \to X/\sigma$ lie on $F_a^{\tilde{R}}$ and $F_b^{\tilde{R}}$. Thus, we have an isomorphism $\tau(U) \cong \tilde{V}$. For $\tau_\pi(v') = v$ and $v' \neq a', b'$, the fibers $F_{v'}^X$ and $F_v^{\tilde{R}}$ are isomorphic. Since X is a K3 surface, π is relatively minimal, any (-1)-components of fibers of $\pi_{\tilde{R}}$ lie on $F_a^{\tilde{R}}, F_b^{\tilde{R}}$. Thus $\tilde{V} \cong V$.

Thus, taking fiber product of $\pi_R \colon R \to \mathbb{P}^1$ by $\tau_\pi \colon \mathbb{P}^1 \to \mathbb{P}^1$, we obtain a fibration which agrees with π over an open set. After resolving singularities and contracting (-1)-curves, we obtain a minimal K3 surface which is birational to X. Since X is minimal, they are isomorphic. □

Remark 3.6.9. Recall that in the case of order 3, Propositions 3.3.9 and 3.3.10 have the hypothesis that σ preserves the zero-section of π. Since for order $p > 3$ we are working under Assumption 3.6.1, this is automatically true for (X, σ).

We can use Proposition 3.6.5 to prove the following.

Proposition 3.6.10. *Let $\pi \colon X \to \mathbb{P}^1$ be an elliptic fibration of type 2 on (X, σ). Then, σ fixes two fibers $F_{a'}$ and $F_{b'}$, and permutes the remaining fibers. The Kodaira types of $F_{a'}$ and $F_{b'}$ are described in Table 15 up to a permutation, and the other fibers are all irreducible (i.e. of Kodaira type I_0, I_1 or II).*

p	$F^X_{a'}$	$F^X_{b'}$
5	III^*, IV	I_{5n}, II^*
	I^*_{5n}	III, IV^*
7	II^*, III	I_{7n}
	IV^*	II, III^*
	I^*_{7n}	IV
11	II	I_{11n}
	III	II^*
	IV	III^*
13	III^*	II
17	IV	III
19	III	II

Tabela 15 – Types of fixed fibers in fibrations of type 2

Demonstração. By Proposition 3.6.8, $\pi\colon X \to \mathbb{P}^1$ is a base change of a rational elliptic surface $\pi_R\colon R \to \mathbb{P}^1$ by a map $\tau\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree p. By Proposition 3.6.5, we know the Kodaira types of fibers of π_R. Thus, we can use Proposition 1.1.23 to determine the fibers of π. □

Let $\pi_X\colon X \to \mathbb{P}^1$ be an elliptic fibration on (X, σ), and $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$ be elliptic fibration it induces on resolution of the quotient X/σ. Any elliptic fibration $\pi\colon X \to \mathbb{P}^1$ (possibly distinct from π_X) induces a pencil of curves Λ on $\tilde{R}$. We can determine Λ by pushing forward the linear system $|F|$ of the fibers of π by the quotient $\tau\colon X \to X/\sigma$, and then applying the pullback by the resolution $\varphi\colon \tilde{R} \to X/\sigma$.

Proposition 3.6.11. *The linear system Λ induced by $\pi\colon X \to \mathbb{P}^1$ is a splitting genus 1 pencil of $\tilde{R}$.*

Demonstração. Suppose π is of type 2. By Proposition 3.6.7, we know that π induces an elliptic fibration $\pi_{\tilde{R}}$. Thus, Λ consists of the system $\{\tilde{F}_v\}_{v\in\mathbb{P}^1}$, where $\tilde{F}_v = \pi_{\tilde{R}}^{-1}(v)$. Consequently, we have $\tilde{F}_v^2 = 0$ and for all but finitely many $v \in \mathbb{P}^1$, $g(\tilde{F}_v) = 1$. By the adjunction formula, $\tilde{F}_v \cdot K_{\tilde{R}} = 0$. Therefore, Λ is a splitting genus 1 pencil. □

Let $\eta_R\colon \tilde{R} \to R$ be the contraction of the (-1)-components on fibers of $\pi_{\tilde{R}}\colon \tilde{R} \to \mathbb{P}^1$. Then, R is a smooth rational surface with a relatively minimal elliptic fibration $\pi_R\colon R \to \mathbb{P}^1$, and by (MIRANDA, 1989, Lemma IV.1.2), there is a birational map $\eta\colon R \to \mathbb{P}^2$.

As a consequence, for any fibration $\pi\colon X \to \mathbb{P}^1$ distinct from π_X, the pencil Λ induces another pencil Γ on $\mathbb{P}^2$ after pushing forward by η_R and η. Since by Proposition 3.6.11 Λ is a genus 1 pencil, the same is true for Γ. We use Γ to deduce an equation for the generic fiber of π. Recall that by Proposition 3.6.8, π is the base change of a rational elliptic surface by a Galois covering $\tau_\pi \colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree p.

Proposition 3.6.12. *Let F_a, F_b be the fibers above the ramification points of τ_π, and $\mathcal{C}_a$, $\mathcal{C}_b$ the induced curves in Γ. Then, we can write the generic fiber of π by the following equation*

$$\pi \colon \mathcal{C}_a(x, y, z) + t^p \mathcal{C}_b(x, y, z) = 0.$$

Demonstração. The linear system Γ is a pencil of genus 1 curves in $\mathbb{P}^2$ generated by $\mathcal{C}_a$ and $\mathcal{C}_b$. Let $\pi' \colon \tilde{R} \to \mathbb{P}^1$ be the elliptic fibration on $\tilde{R}$ induced by Λ. Then, for all but finitely many $t \in \mathbb{P}^1$, the fiber $F_t = (\pi')^{-1}(t)$ is isomorphic to $\mathcal{C}_a(x, y, z) + t\mathcal{C}_b(x, y, z) = 0$. By a change of coordinates, we can suppose that τ_π is given by the map $t \mapsto t^p$. Thus, applying the base change by τ_π, we obtain the equation for the generic fiber of π. □

Conclusão

In the last part of this thesis, we present further questions and topics of future research that follow from the content of the two main chapters.

Chapter 2. There are different ways of generalizing Nagao's conjecture to the ranks of Jacobian varieties of hyperelliptic curves over $k(T)$ (see (WONG, 2001), (HINDRY; PACHECO, 2005), (HAMMONDS et al., 2019)). Specifically, in (HAMMONDS et al., 2019) these formulas were used to determine the ranks for specific families of hyperelliptic curves.

Let χ be a hyperelliptic curve over $k(T)$ given by $y^2 = f(x, T)$, where $\deg_T(f) \leq 2$ and $\deg_x(f) \geq 5$. The Kodaira–Néron model of χ is a smooth projective surface X endowed with a conic bundle $\varphi\colon X \to \mathbb{P}^1$ and a fibration $\pi\colon X \to \mathbb{P}^1$ in curves of genus $g > 1$. Is there a way to use these geometric structures to reinterpret and generalize the results of (HAMMONDS et al., 2019)?

Chapter 3. Let X be a K3 surface and $\sigma \in \text{Aut}(X)$ a non-symplectic automorphism. In Chapter 3, we add several conditions to the pair (X, σ) to apply our method. The natural next step is to figure out which conditions can be relaxed.

1. Propositions 3.3.9, 3.3.10 and 3.3.12 require the hypothesis that σ preserves the zero-section of $\pi\colon X \to \mathbb{P}^1$. If this is not true, then π determines a genus 1 fibration on the rational surface $\tilde{R}$ (i.e. an elliptic fibration without section). Rational genus 1 fibrations are constructed as the resolution of a Halphen pencil on $\mathbb{P}^2$ ((COSSEC; DOLGACHEV, 1989, Theorem 5.6.1)). Can this construction be used to obtain equations for the generic fibers of elliptic fibrations on X?
2. The results of Sections 3.3.3 and 3.3.4 rely on Assumption 3.3.14 that X admits an elliptic fibration of type 2. By Proposition 3.2.13 this is not true for pairs (X, σ) where σ fixes a curve of genus $g \geq 2$. For K3 surfaces with a non-symplectic involution ι, these cases were studied in (GARBAGNATI; SALGADO, 2020) when ι also fixes a rational curve, and in (COMPARIN et al., 2023) when ι only fixes one curve of genus $g \geq 2$. What can be said for order 3?

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